PhD Comprehensive Exam (Algebra)
Department of Mathematics
8 October 2025, 2:00 - 5:00 pm in MC 108

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given: aim for complete solutions.
3. You should attempt at least one question from each topic.
4. Justify all your answers.

Linear Algebra

1. Let n be a positive integer and N be an upper-triangular $n \times n$-matrix over $\mathbb{C}$ satisfying $N^n = 0$. Show that $\det(f(N)) = f(0)^n$ for every polynomial $f \in \mathbb{C}[x]$.
2. Let M be a 6×6-matrix over $\mathbb{C}$ satisfying

$$\dim(\ker(M^3)) = 2 \quad \text{and} \quad \dim(\ker((M-1)^2)) = 3.$$

   What are the possible Jordan normal forms of M?

Groups

3. Prove that, if H is a subgroup of a group G with index $|G : H| = n < \infty$, then there is a normal subgroup N of G with $N \leq H$ such that $|G : N|$ divides $n!$.
4. Let G be a group with $|G| = 12$. Prove that G is a semidirect product.

Rings and Modules

5. Let R be a commutative ring with identity, and let M be a (left) R-module. Prove that $\mathrm{Hom}_R(R, M) \cong M$ as R-modules. Point out where the commutativity of R is used.
6. Let m and n be positive integers with greatest common divisor d. Prove that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/d\mathbb{Z}$.

Fields

7. Construct the finite field $\mathbb{F}_{2^4} = \mathbb{F}_{16}$ and a generator of the multiplicative group $\mathbb{F}_{2^4}^{\times}$. Explain why your constructions are valid.
8. Suppose $E/\mathbb{Q}$ and $F/\mathbb{Q}$ are Galois extensions with group $\mathbb{F}_{2^2} \simeq \mathbb{F}_2^2$.
   (a) Explicitly determine all possible isomorphism classes for the Galois group G of $E \cap F$ over $\mathbb{Q}$.
   (b) For each class G, give an explicit pair (E, F) where $E \cap F$ is Galois over $\mathbb{Q}$ with group G.