THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 14 2025, 2 - 5 PM

Instructions: This exam contains 8 problems. Solve as many problems as you can. Carefully justify your answers. Aim for complete solutions (there will be little or no credit for partial answers).

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1. Compute
$$\lim_{\mathbb{C}\ni z\to 0} \frac{\sin z}{z}.$$
Justify your answer.

2. Prove that all the roots of $p(z) = z^7 - 5z^3 + 12$ lie between the circle $|z| = 1$ and $|z| = 2$.

3. Evaluate
$$\int_{-\infty}^{\infty} \frac{dx}{(1+x+x^2)^2} \quad .$$

4. Suppose that $g(z)$ is a function that is holomorphic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ and continuous on its closure, $\overline{\mathbb{D}}$. Assume that $\operatorname{Im} g(z) \equiv 0$ on the unit circle $\partial\mathbb{D}$. Prove that $g(z)$ is a constant function.

5. Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove:
$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \ldots + na_n}{n} = 0.$$

6. Let C be the unit circle centered at the origin, oriented counterclockwise, in the xy-plane. Calculate the line integral
$$\int_C \Big((1 + \frac{\arctan x}{1+x^4} + y^3)dx + (y\cos(y^3) + xy^2)dy\Big).$$

7. Let (X, d), (Y, ρ) be metric spaces. Let $(f_n)_{n\geq 1}$ be a sequence of mappings $X \to Y$ that converges uniformly on X to $f : X \to Y$. Suppose for each positive integer n, there is a real number M_n such that $\rho(f_n(x), f_n(y)) \leq M_n$ for all $x, y \in X$.

   Prove: there is a real number M such that $\rho(f_n(x), f_n(y)) \leq M$ for all $x, y \in X$ and all $n \in \mathbb{N}$.

8. Show that there is a unique continuous function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \frac{1}{2}e^{-x}\left[\cos(x^2) + \int_0^1 e^{-y} f(y)dy\right].$$

Hint: use the Contraction Mapping Theorem.