Instructions: Answer completely as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

(1) Find all ring homomorphisms \( f: \mathbb{Z} \to \mathbb{Z}/18\mathbb{Z} \).

(2) For \( n \geq 2 \), characterize the \( n \times n \) matrices over \( \mathbb{C} \) which commute only with diagonalizable matrices.

(3) Show that any group of order 10 is either a cyclic group or a dihedral group.

(4) (a) Let \( G \) be a group. Show that the conjugation homomorphism \( c: G \to \text{Aut}(G) \) is injective if and only if the centre of \( G \) is trivial.

(b) If \( G \) is simple and nonabelian, is \( c \) necessarily an isomorphism? Prove or give a counterexample.

(5) Suppose that \( A \) and \( B \) are \( 4 \times 4 \) matrices over \( \mathbb{C} \) with the same minimal polynomial, characteristic polynomial, and at least two distinct eigenvalues. Prove that \( A \) and \( B \) are similar. Find an example of two \( 5 \times 5 \) matrices over \( \mathbb{C} \) with the same properties that are not similar.

(6) Let \( R \) be an integral domain. For an \( R \)-module \( M \), let \( M^* = \text{Hom}_R(M, R) \).

(a) Verify that the function \( i_M: M \to M^{**} \) given by
\[
i_M(m)(f) = f(m)
\]
for \( m \in M \) and \( f \in M^* \) is a \( R \)-module homomorphism, for any \( M \).

(b) Show that \( i_M \) is injective if and only if \( M \) is torsion-free. (Assume \( M \) is finitely generated here.)

(c) If \( R \) is a PID, show that \( i_M \) is an isomorphism if \( M \) is torsion-free.

(d) Give an example of a ring \( R \) and \( R \)-module \( M \) for which \( i_M = 0 \).

(e) Give an example of a ring \( R \) and \( R \)-module \( M \) for which \( i_M \) is injective but not surjective.

(7) Show that, for positive integers \( m, n \),
\[
\mathbb{Z}/m\mathbb{Z} \bigoplus \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}
\]
as abelian groups, where \( d = \gcd(m, n) \).

(8) Let \( G \) be a finite group of order 504 = \( 2^3 \cdot 3^2 \cdot 7 \).

(a) Show that \( G \) cannot be isomorphic to a subgroup of the alternating group \( A_7 \).

(b) If \( G \) is simple, determine the number of Sylow 3-subgroups.

(9) Let \( E \) be a splitting field of \( x^3 - 2 \) over the rationals \( \mathbb{Q} \) and assume that \( E \) is a subfield of \( \mathbb{C} \). Let \( F = E \cap \mathbb{R} \) be the real subfield and note that \( F = \mathbb{Q}(\sqrt[3]{2}) \).

(a) Show that \( \text{Gal}(E/\mathbb{Q}) \) contains an element \( \sigma \) with the property that all elements of \( F \) fixed by \( \sigma \) are rational.

(b) Let \( a \in F \) and suppose \( a^3 \in \mathbb{Q} \). Show that one of \( a, a\sqrt[3]{2} \) or \( a\sqrt[3]{4} \) is contained in \( \mathbb{Q} \).

(c) Prove that \( \sqrt[3]{3} \notin E \).