THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENT OF MATHEMATICS

Ph. D. Comprehensive Exam (Analysis)

October 2010 3 hours

Answer completely as many questions as you are able. More credit will be given for several complete solutions than for many partial solutions.

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1. Find a Möbius transformation mapping the half-plane $\{z : \operatorname{Re} z < 1\}$ onto $\{z : |z-1| > 2\}$.

2. Suppose that $g : [0,1] \longrightarrow \mathbb{C}$ is continuous. Prove that g is uniformly continuous.

3. Classify the singularity of $\dfrac{z^2(z-1)}{(1-\cos z)\log(1+z)}$ at $z = 0$.

4. Let (X, d) be a metric space and a be a point of X. Define

$$\rho(x,y) = \begin{cases} d(x,a) + d(a,y) & x \neq y \\ 0 & x = y. \end{cases}$$

   (a) Show that (X, ρ) is a metric space.

   (b) Show that if a subset of X is open in (X, d) then it is open in (X, ρ).

   (c) Give an example of a metric space (X, d) and a point $a \in X$ such that the topologies on (X, d) and (X, ρ) coincide but are not just the discrete topology.

5. Let f be an **even** meromorphic function, that is to say, let f be a meromorphic function such that $f(-z) = f(z)$ for all z, and suppose that f has a pole at 0. Show that the residue of f at 0 is equal to 0.

6. Suppose $\phi_n : \mathbb{R} \longrightarrow (0, \infty)$ satisfy

$$\int_{\mathbb{R}} \phi_n(t)dt = 1 \ \text{ for } \ n = 1, 2, \ldots; \quad \text{and} \quad \lim_{n\to\infty} \int_{|t|>\delta} \phi_n(t)dt = 0 \ \text{ for every } \ \delta > 0.$$

   If f is a bounded function on $\mathbb{R}$ which is continuous at x prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(x-t)f(t)dt = f(x).$$

7. Evaluate
$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3}\, dx.$$

8. In a game of hide and seek on the complex plane the hider is hiding in a tree at the origin. The seeker runs counterclockwise along the unit circle from 1 to -1 at unit speed. When the seeker reaches $e^{i\pi/4}$, the hider leaves the tree and runs at a constant speed to 1 always keeping the tree directly between himself and the seeker. The hider arrives at 1 at the same time that the seeker arrives at -1. What path does the hider follow? (Hint: Express the position of the hider in polar form, find the argument, and use constant speed to determine the modulus.)

9. Apply the maximum principle to find the smallest number A for which the inequality $|\sin z| \leq A|z|$ is satisfied in $\{z : |z| \leq 1\}$.