THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination
Algebra

May 2010 3 hours

Instructions: Completely answer as many questions as you are able. More credit will be given for a complete solution than for several partial solutions. **Explain all answers fully.**

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1. (a) Up to similarity, list all 4×4 matrices in $M_4(\mathbb{C})$ which have characteristic polynomial $\lambda(\lambda - 1)^3$.

   (b) For each matrix you gave above, write down its minimal polynomial.

   (c) Let $A \in M_n(\mathbb{C})$. Prove that A is similar to a diagonal matrix if and only if its minimal polynomial has distinct roots.

2. Let V be a finite-dimensional inner product space over $\mathbb{C}$ and let T be a linear operator on V.

   (a) Define the **adjoint** of T and define what it means for T to be **self-adjoint**.

   Recall that T is said to be **normal** if it commutes with its adjoint.

   (b) Prove that if T is normal then T and its adjoint T^* have the same kernel.

   (c) Prove that if T is normal and $T = T^2$ then T is self-adjoint.

   (d) Prove that if T is normal and nilpotent, then $T = 0$.

3. Let S and T be commuting operators on a finite-dimensional vector space V over an algebraically closed field k.

   (a) Prove that S and T have a common eigenvector. (You may use the fact that any operator on a vector space over k has at least one eigenvector.)

   (b) If V has a basis of eigenvectors of S and a basis of eigenvectors of T, show that it has a basis consisting of vectors that are eigenvectors for both S and T.

   (c) What does (b) say about matrices?

4. Let p and q be distinct primes with $p < q$ and $q \not\equiv 1 \pmod{p}$. Let G be a group of order pq. Prove that G is cyclic.

5. (a) Define what it means for a group G to be **simple**.

   (b) Prove that if $|G| = 30$, then G is not simple.

6. Let S_4 be the symmetric group on 4 letters. Prove or disprove: every two subgroups of S_4 of order 4 are conjugate.

7. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

   (a) Compute the Galois group of K over $\mathbb{Q}$. Explain fully.

   (b) List the distinct subfields of K.

   (c) Is the extension $K/\mathbb{Q}$ Galois? If so, indicate the Galois correspondence between (a) and (b).

8. (a) Define what it means for an algebraic extension $k \subseteq K$ of fields to be **normal**.

   (b) Find an algebraic extension of $\mathbb{Q}$ which is not normal. Explain fully.

9. Let R be a commutative ring with 1, N a *nilpotent* ideal of R, and $\pi\colon R \to R/N$ the quotient map.

   (a) Prove that if $\pi(r)$ is a unit (invertible element) in R/N, then r is a unit in R.

   Recall that $GL_n(R)$ denotes the group of invertible $n \times n$ matrices over R.

   (b) Prove that the induced map from $GL_n(R)$ to $GL_n(R/N)$ is surjective.

10. Let k be a field.

    (a) Prove that the polynomial ring $k[t]$ is a principal ideal domain.

    (b) Suppose that $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals in a principal ideal domain R. Prove that there is a number N such that $I_N = I_{N+1} = \cdots$.

    (c) Prove that every element of a principal ideal domain R is a product of irreducible elements.

11. Let R be an integral domain with field of fractions F.

    (a) Define what it means for an element a of F to be **integral** over R.

    (b) Define what it means for R to be **integrally closed**.

    (c) Show that a unique factorization domain is integrally closed.