THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENT OF MATHEMATICS

Ph. D. Comprehensive Exam (Analysis)

May 2010 3 hours

Answer completely as many questions as you are able. More credit will be given for several complete solutions than for many partial solutions.

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1. Suppose that f is holomorphic for $|z| < 1$. Suppose that $|f(z)| \le 1$ for all $|z| < 1$, and

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0.$$

Prove that $|f(z)| \le |z|^k$, for all $|z| < 1$.

2. Let γ be the positively oriented circle $|z| = 1/2$. Evaluate

$$\int_\gamma \frac{e^{1/z}}{1-z} dz.$$

3. Consider the linear fractional transformation $f(z) = \frac{az+b}{cz+d}$ with $ad - bc \neq 0$ as a map on the extended complex plane $\mathbb{C} \cup \{\infty\}$, i.e. on the Riemann sphere. Show that f maps circles in the extended complex plane to circles.
Hint: First prove that f can be written as $f(z) = A + \frac{B}{z+C}$.

4. Let $P(z)$ be a polynomial in z. Suppose that all zeros of $P(z)$ are contained in the upper half plane. Prove that the zeros of $P'(z)$ are also contained in the upper half plane.
Hint: Consider $\dfrac{P'}{P}$ (logarithmic differentiation).

5. Suppose

$$f(z) = az^2 + bz\overline{z} + c\overline{z}^2$$

where a, b, and c are fixed complex numbers.

(a) Show that $f(z)$ is complex differentiable at z if and only if $bz + 2c\overline{z} = 0$

(b) Where is $f(z)$ analytic?

Justify your answers.

6. (a) Show that the area of a planar region delimited by a closed simple curve C is given by $\frac{1}{2}\int_C x\,dy - y\,dx$.

   (b) Compute $\int_C (2xy - x^2)\,dx + (x + y^2)\,dy$, where C is the the boundary of the bounded region delimited by the graphs of $y = x^2$ and $y^2 = x$.

7. (a) Show that every subspace of a separable metric space is separable.

   (b) Let X be a separable metric space and let $Y \subset X$ be any subspace. Given $N \in \mathbb{N}$, construct a sequence $\{a_k\}$ where each $a_k = (a_{k,1}, a_{k,2}, \dots, a_{k,N}) \in Y^N$, with the property that, given any $y \in Y^N$, there is a subsequence $\{a_{k_i}\}$ converging to y.

8. Let (X, d) be a complete metric space. Show that a contraction $f : X \to \mathbb{R}$ is necessarily continuous and has precisely one fixed point. Recall that f is a contraction if there is a constant $0 < C < 1$ such that

$$d(f(x), f(y)) \le C d(x, y) \quad \text{for all} \ \ x, y \in X.$$

9. Let $X = C[0, 1]$ with the topology of uniform convergence.

   (a) Is the subspace $\mathcal{P}$ of polynomials open in X?

   (b) Is $\mathcal{P}$ closed?

   Justify your answers.

10. *Helly's selection principle* states that given a sequence (f_n) of nondecreasing functions $f_n : [0, 1] \to [a, b]$, there exists a subsequence (f_{n_k}) and a function $F : [0, 1] \to [a, b]$ such that for any $x \in [0, 1]$, $\lim_{k\to\infty} f_{n_k}(x) = F(x)$. The proof is divided into three steps:

    (a) Show that we can find a subsequence (f_{n_k}) that converges pointwise to a nondecreasing function G defined on all rational points $\{r_1, r_2, r_3, \dots\}$ of $[0, 1]$.

    (b) Define $H : [0, 1] \to [a, b]$ by setting

$$H(x) = \lim_{\substack{r\to x^- \\ r\in\mathbb{Q}\cap[0,1]}} G(r)$$

    Show that H is the limit of (f_{n_k}) at each continuity point of H.

    (c) Now, recall that a nondecreasing real valued function of a real variable has at most countably many discontinuity points. Use a diagonal argument to find a subsequence of (f_{n_k}) that converges everywhere on $[0, 1]$ to some function F.