1. Suppose that $f$ is holomorphic for $|z| < 1$. Suppose that $|f(z)| \leq 1$ for all $|z| < 1$, and 

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0.$$ 

Prove that $|f(z)| \leq |z|^k$, for all $|z| < 1$.

2. Let $\gamma$ be the positively oriented circle $|z| = 1/2$. Evaluate 

$$\int_\gamma \frac{e^{1/z}}{1 - z} dz.$$

3. Consider the linear fractional transformation $f(z) = \frac{az + b}{cz + d}$ with $ad - bc \neq 0$ as a map on the extended complex plane $\mathbb{C} \cup \{\infty\}$, i.e. on the Riemann sphere. Show that $f$ maps circles in the extended complex plane to circles. 

*Hint:* First prove that $f$ can be written as $f(z) = A + \frac{B}{z + C}$.

4. Let $P(z)$ be a polynomial in $z$. Suppose that all zeros of $P(z)$ are contained in the upper half plane. Prove that the zeros of $P'(z)$ are also contained in the upper half plane. 

*Hint:* Consider $\frac{P'}{P}$ (logarithmic differentiation).

5. Suppose 

$$f(z) = az^2 + b\overline{z} + c\overline{z}^2$$

where $a, b,$ and $c$ are fixed complex numbers.

(a) Show that $f(z)$ is complex differentiable at $z$ if and only if $bz + 2c\overline{z} = 0$

(b) Where is $f(z)$ analytic? 

Justify your answers.
6. (a) Show that the area of a planar region delimited by a closed simple curve \( C \) is given by 
\[
\frac{1}{2} \int_C x \, dy - y \, dx.
\]
(b) Compute \( \int_C (2xy - x^2) \, dx + (x + y^2) \, dy \), where \( C \) is the boundary of the bounded region delimited by the graphs of \( y = x^2 \) and \( y^2 = x \).

7. (a) Show that every subspace of a separable metric space is separable.

(b) Let \( X \) be a separable metric space and let \( Y \subset X \) be any subspace. Given \( N \in \mathbb{N} \), construct a sequence \( \{a_k\} \) where each \( a_k = (a_{k,1}, a_{k,2}, \ldots, a_{k,N}) \in Y^N \), with the property that, given any \( y \in Y^N \), there is a subsequence \( \{a_{k_i}\} \) converging to \( y \).

8. Let \( (X,d) \) be a complete metric space. Show that a contraction \( f : X \to \mathbb{R} \) is necessarily continuous and has precisely one fixed point. Recall that \( f \) is a contraction if there is a constant \( 0 < C < 1 \) such that 
\[
d(f(x), f(y)) \leq Cd(x, y) \quad \text{for all } x, y \in X.
\]

9. Let \( X = C[0,1] \) with the topology of uniform convergence.

(a) Is the subspace \( \mathcal{P} \) of polynomials open in \( X \)?

(b) Is \( \mathcal{P} \) closed?

Justify your answers.

10. *Helly’s selection principle* states that given a sequence \( (f_n) \) of nondecreasing functions \( f_n : [0,1] \to [a,b] \), there exists a subsequence \( (f_{n_k}) \) and a function \( F : [0,1] \to [a,b] \) such that for any \( x \in [0,1] \), \( \lim_{k \to \infty} f_{n_k}(x) = F(x) \). The proof is divided into three steps:

(a) Show that we can find a subsequence \( (f_{n_k}) \) that converges pointwise to a nondecreasing function \( G \) defined on all rational points \( \{r_1, r_2, r_3, \ldots\} \) of \([0,1] \).

(b) Define \( H : [0,1] \to [a,b] \) by setting
\[
H(x) = \lim_{\substack{r \to x \\scriptscriptstyle r \in \mathbb{Q} \cap [0,1]}} G(r)
\]

Show that \( H \) is the limit of \( (f_{n_k}) \) at each continuity point of \( H \).

(c) Now, recall that a nondecreasing real valued function of a real variable has at most countably many discontinuity points. Use a diagonal argument to find a subsequence of \( (f_{n_k}) \) that converges everywhere on \([0,1] \) to some function \( F \).