ALGEBRA QUALIFYING EXAM

UWO, SPRING 2011

Time: 180min; no aids allowed. **All answers must be justified** unless stated otherwise. They will be graded according to correctness, completeness, relevance and clarity of presentation.

$\mathbb{C}^{n\times n}$ denotes the ring of complex $n \times n$-matrices.

(1) Let $A \in \mathbb{C}^{n\times n}$. How can one read off the degree of the minimal polynomial of A from the Jordan canonical form of A?

(2) Consider a (two-sided) ideal $I \in \mathbb{C}^{n\times n}$. Show that $I = 0$ or $I = \mathbb{C}^{n\times n}$.

(3) Let V be a finite-dimensional $\mathbb{Q}$-vector space, and $A: V \to V$ a linear map such that $A^5 = \mathrm{id}_V$. Assume further that A has no fixed point apart from $0 \in V$. Prove that $\dim V$ is divisible by 4.

(4) Determine all n such that the ring $\mathbb{Z}_n$ has exactly 12 invertible elements.

(5) Give an example of a free module M over some commutative ring R and a submodule $N \subset M$ that is torsion-free, but not free. (Justify why N is not free.)

(6) Let R be an integral domain.
   (a) Define the field of fractions $\mathrm{Quot}(R)$ of R and the canonical morphism $\varphi_R: R \to \mathrm{Quot}(R)$.
   (b) When is φ_R an isomorphism? (Justify!)

(7) Let G be a group and $H \triangleleft G$. Show that if H and G/H are soluble, then so is G.

(8) Show that any group of order 91 is cyclic.

(9) Determine the number of conjugacy classes in the symmetric group S_5 and the number of elements in each class.

(10) Let F be a field and G be a finite subgroup of the multiplicative group $F \setminus \{0\}$. Show that G is cyclic.

(11) State and prove Eisenstein's irreducibility criterion.

(12) Let p be a prime, and let m and n be two positive integers such that m divides n.
   (a) Explain why $\mathbb{F}_{p^m}$ is a subfield of $\mathbb{F}_{p^n}$.
   (b) Compute $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_{p^m})$.