THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 4, 2011 *3 hours*

Instructions: Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

1. Let

$$T_n = \left\{(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in [0,\infty)^{2n} : \sum_{k=1}^{n} \max(x_n, y_n)^2 \leq 1\right\}$$

and let α_n be the $2n$-dimensional volume of T_n. State and prove a simple formula for α_n in terms of n.

2. Let h be a real-valued, continuous, non-negative, non-increasing function on $[0,\infty)$. For $n = 1, 2, 3, \ldots$, define h_n by

$$h_n(x) = n\int_x^{x+(1/n)} h(t)\,dt.$$

Prove that $h_1, h_2, h_3, \ldots$ is a non-decreasing sequence of differentiable, non-negative, non-increasing functions that converges pointwise to h.

3. Let X be a bounded subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ be a continuous function. Suppose that for each $y \in \mathbb{R}^n \setminus X$ there exists a $\delta_y > 0$ and a constant B_y such that $|f(x)| \leq B_y$ for all $x \in X$ such that $|x - y| < \delta_y$. Prove that f is bounded.

4. Let (T, d) and (Y, ρ) be complete metric spaces. Suppose S is a subset of T, $f : S \to Y$ is uniformly continuous, and t is in the closure of S but not in S. Prove that f extends to a continuous function $g : S \cup \{t\} \to Y$. That is, prove that there exists $y \in Y$ such that

$$g(s) = \begin{cases} f(s), & s \in S \\ y, & s = t. \end{cases}$$

is continuous at t.

5. Expand $\dfrac{1}{(z-1)(z-2)}$ in a Laurent series centered at $z = 0$ and converging in the annulus $1 < |z| < 2$.

6. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$.

7. Fix $n \geq 1$, $r > 0$, and $\lambda = \rho e^{i\phi}$. What is the maximum modulus of $z^n + \lambda$ over the disc $|z| \leq r$? Where does $z^n + \lambda$ attain its maximum modulus over the disc ?

8. Show that the image of a nonconstant entire function is dense in $\mathbb{C}$.