UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

September 2012 3 hours

Instructions: Answer completely as many questions as you can. More credit will be given for a complete solution than for several partial solutions. Grading criteria inlude correctness, completeness, relevance and quality of exposition.

(1) Let V be a finite-dimensional, complex vector space with a Hermitian inner product. Let $T : V \to V$ be a self-adjoint linear transformation.
  (a) Show that every eigenvalue of T is real.
  (b) Show that if u and v are eigenvectors for distinct eigenvalues, then u and v are orthogonal.

(2) You are given an $n \times n$ matrix A. Its minimal polynomial equals t^4, and its null space is 4-dimensional. What are the possible values of n?

(3) Let V be a 3-dimensional vector space over the field $\mathbb{F}_5$. How many linear automorphisms does V have?

(4) Let G be a group. Call two elements g, $h \in G$ equivalent if $\langle g \rangle = \langle h \rangle$, that is, if they generate the same cyclic subgroup of G. Show that the corresponding equivalence classes are finite.

(5) Let G be a finite group with exactly two conjugacy classes. Show that G is isomorphic to $\mathbb{Z}_2$.

(6) Show that all groups of order 45 are abelian.

(7) Show that if $f : R \to S$ is a (unital) ring homomorphism and I is a prime ideal of S, then $f^{-1}(I)$ is a prime ideal of R.

(8) We consider the ring $R = \mathbb{Z}[x, y]$.
  (a) Prove that the ideal $I = (x - 1, y)$ in R is prime.
  (b) Is I a maximal ideal? Prove or disprove.
  (c) Give an example of an ideal in R that is not prime. (Justify your choice.)

(9) List all $\mathbb{Z}$-modules M for which there exists an exact sequence of the form

$$0 \longrightarrow \mathbb{Z}_{50} \longrightarrow M \longrightarrow \mathbb{Z}_{12} \longrightarrow 0.$$

(Justify!)

(10) For a fixed prime power $q = p^n$, let $\sigma : \mathbb{F}_q \to \mathbb{F}_q$ be the Frobenius homomorphism (given by $\sigma(x) = x^p$ for all $x \in \mathbb{F}_q$). Compute the minimal and characteristic polynomials of σ.

(11) Let $f(X) = X^3 - 2$, and let $E/\mathbb{Q}$ be the splitting field of $f(X)$. Compute the Galois group of $E/\mathbb{Q}$ and determine the fixed field E^H for each subgroup $H < \mathrm{Gal}(E/\mathbb{Q})$ of order 3.

(12) Let E/F be a normal field extension and assume that any non-constant $f(X) \in F[X]$ has a root in E. Show that E is algebraically closed.