THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENT OF MATHEMATICS

Ph. D. Comprehensive Exam (Analysis)

September 27, 2012 13:30 p.m. – 16:30 p.m.

Answer completely as many questions as you are able. More credit will be given for several complete solutions than for many partial solutions.

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1. Solve the boundary value problem

$$xf'' = 4f' - 25x^9 f, \quad f(0) = 0, \; f(1) = 1$$

by making the substitution $x^5 = t$.

2. Given a sequence of continuous functions $\phi_n : \mathbb{R} \to [0, \infty)$ satisfying

$$\int_{\mathbb{R}} \phi_n(t)\,dt = 1 \quad \text{and} \quad \lim_{n\to\infty} \int_{|t|>\delta} \phi_n(t)\,dt = 0, \text{ for all } \delta > 0$$

show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(x-t) f(t)\,dt = f(x)$$

whenever $f : \mathbb{R} \to \mathbb{R}$ is continuous at x and bounded on $\mathbb{R}$.

3. Let $(y_n)_{n=1}^{\infty}$ be a sequence of real numbers and define $f : \mathbb{R} \to \mathbb{R}$ by setting

$$f(x) = \inf_{n\in\mathbb{Z}_+} n|x - y_n|\,.$$

(a) Show that if (y_n) has no accumulation point then f is continuous.

(b) Find a sequence (y_n) for which f is not continuous. Justify your answer.

4. Let C be the set of continuous real-valued functions on $[0, 1]$. Given $f, g \in C$, define

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)| \quad \text{and} \quad \rho(f, g) = \int_0^1 |f(t) - g(t)|\,dt\,.$$

(a) Show that d and ρ are metrics on C.

(b) Prove that (C, d) is complete.

(c) Show that (C, ρ) is not complete.

5. Let γ be the circle with radius 2 centered at 1 traversed one time counterclockwise. Evaluate the integrals:

(a) $\displaystyle\int_\gamma \frac{e^{2z}}{(1+z^2)^2}\,dz$

(b) $\displaystyle\int_\gamma \frac{\sin(\pi z)}{z^2-2z}\,dz\,.$

6. How many solutions, counted with multiplicities, does the equation $e^{-z} = 2z^3 + 3z + 1$ have in the disc $|z| < 2$?

7. Evaluate $\displaystyle\int_0^\infty \frac{dx}{1+x^4}\,.$

8. Find three different Laurent series for $f(z) = \dfrac{1}{z-2} - \dfrac{1}{z} + \dfrac{1}{(z+1)^2}$ about $z_0 = 1$ and state their regions of convergence.