WESTERN UNIVERSITY
Department of Mathematics
Ph. D. Comprehensive Exam (Algebra)

May 2012 3 hours

Instructions: Give complete justifications for your answers, unless otherwise specified. Complete an many questions as you can. More credit will be given for complete solutions than for several partial solutions.

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1. (10 points) Let R be a commutative ring with $1 \neq 0$. Suppose that R has only two ideals. Show that R is a field.

2. (10 points) Denote by $M_{2\times 2}(\mathbb{C})$ the 4 dimensional complex vector space of 2×2 matrices with complex coefficents. Let V be a $\mathbb{C}$-subspace of $M_{2\times 2}(\mathbb{C})$. Suppose that V consists of commuting matrices. Show that $\dim V \leq 2$.

3. (10 points) Consider a surjective homomorphism
$$q : G' \twoheadrightarrow G$$
of groups. Suppose that q has a section, i.e there is a group homomorphism $s : G \to G'$ such that $q \circ s = 1_G$. Here 1_G is the identity homomorphism. Show that, G' is a semi-direct product. More precisely show that
$$G' \cong \ker(q) \rtimes G.$$

4. (10 points) (a) (4 points) Suppose that we have an algebraic field extension L/K. Define what it means for L/K to be normal.
   (b) (3 points) Give an example of a finite extension that is not normal. (Remember to justify your example.)
   (c) (4 points) Prove or disprove : Suppose that L/K and M/L are finite normal extensions. Then M/K is normal.

5. (10 points) Let $\mathbb{F}$ be a finite field with p elements, with p prime. Choose $\alpha \in \mathbb{F}$ with $\alpha \neq 0$. Consider the polynomial
$$X^p - X + \alpha.$$
(Such a polynomial is called an Artin-Schreier polynomial, although this fact is not important for the question.) Show that this polynomial is irreducible. (Hint : observe that the polynomial is invariant under the transformation $X \mapsto X + b$.)

6. (10 points) (a) (3 points) Produce a table of abelian groups of order 8 so that every abelian group of order 8 is isomorphic to exactly one group in your table. (You do not need to justify your answer.)

(b) (7 points) How many subgroups of order 4 does the group $\mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2$ have. You should fully and CLEARLY justify your answer. (You will be marked on clarity of your solution!)

7. (10 points) Does there exist a group of order 5^3 whose center has order 5^2? (Note : Yes or No anwers will receive no credit. Make sure that you give adequate justification for your answer.)

8. (10 points) (a) (2 points) Let $\mathbb{F}$ be finite field and denote by $\mathbb{F}^*$ the group of non-zero elements of $\mathbb{F}$ under multiplication. Show that $\mathbb{F}^*$ is cyclic. (Hint : Let d be maximal so that there is an $x \in \mathbb{F}^*$ with $\text{ord}(x) = d$. Here $\text{ord}(x)$ denotes the order of the element x in the group $\mathbb{F}^*$. Consider the equation $X^d - 1$ in $\mathbb{F}$.)

   (b) (4 points) Consider the finite field $\mathbb{F}_9$ of order 9. Find all generators for the cyclic group $\mathbb{F}_9^*$.

   (c) (4 points) Use the first part to show that the field extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ is a separable extension. Here $\mathbb{F}_{p^n}$ is the finite field with p^n elements.

9. (8 points) Let V be a finite dimensional vector space over $\mathbb{Q}$. Consider a linear operator $T : V \to V$ with characteristic polynomial $c(x) \in \mathbb{Q}[x]$. Let $f(x) \in \mathbb{Q}[x]$ be coprime to $c(x)$. Show that the operator $f(T)$ is invertible.

10. (7 points) Prove or disprove : Every finite group G is isomorphic to a subgroup of a dihedral group D_{2n} for some n. Recall : that D_{2n} is the group of symmetries of a regular n-gon. (Remember to fully justify your answer.)

11. (5 points) Suppose that G is a finite set with an associative binary operation denoted by $\circ$. Suppose that for all $a, b \in G$ the equations

$$a \circ x = b \qquad x \circ a = b$$

have at least one solution x in G. Show that G, with multiplication defined by $\circ$, is a group.