1. (10 points) Let \( R \) be a commutative ring with \( 1 \neq 0 \). Suppose that \( R \) has only two ideals. Show that \( R \) is a field.

2. (10 points) Denote by \( M_{2 \times 2}(\mathbb{C}) \) the 4 dimensional complex vector space of \( 2 \times 2 \) matrices with complex coefficients. Let \( V \) be a \( \mathbb{C} \)-subspace of \( M_{2 \times 2}(\mathbb{C}) \). Suppose that \( V \) consists of commuting matrices. Show that \( \dim V \leq 2 \).

3. (10 points) Consider a surjective homomorphism \( q : G' \rightarrow G \) of groups. Suppose that \( q \) has a section, i.e. there is a group homomorphism \( s : G \rightarrow G' \) such that \( q \circ s = 1_G \). Here \( 1_G \) is the identity homomorphism. Show that, \( G' \) is a semi-direct product. More precisely show that 

\[
G' \cong \ker(q) \rtimes G.
\]

4. (10 points) (a) (4 points) Suppose that we have an algebraic field extension \( L/K \). Define what it means for \( L/K \) to be normal.

(b) (3 points) Give an example of a finite extension that is not normal. (Remember to justify your example.)

(c) (4 points) Prove or disprove: Suppose that \( L/K \) and \( M/L \) are finite normal extensions. Then \( M/K \) is normal.

5. (10 points) Let \( \mathbb{F} \) be a finite field with \( p \) elements, with \( p \) prime. Choose \( \alpha \in \mathbb{F} \) with \( \alpha \neq 0 \). Consider the polynomial

\[
X^p - X + \alpha.
\]

(Such a polynomial is called an Artin-Schreier polynomial, although this fact is not important for the question.) Show that this polynomial is irreducible. (Hint: observe that the polynomial is invariant under the transformation \( X \mapsto X + b \).)

6. (10 points) (a) (3 points) Produce a table of abelian groups of order 8 so that every abelian group of order 8 is isomorphic to exactly one group in your table. (You do not need to justify your answer.)
(b) (7 points) How many subgroups of order 4 does the group \( \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2 \) have. You should fully and CLEARLY justify your answer. (You will be marked on clarity of your solution!)

7. (10 points) Does there exist a group of order \( 5^3 \) whose center has order \( 5^2 \)? (Note : Yes or No answers will receive no credit. Make sure that you give adequate justification for your answer.)

8. (10 points) (a) (2 points) Let \( \mathbb{F} \) be finite field and denote by \( \mathbb{F}^* \) the group of non-zero elements of \( \mathbb{F} \) under multiplication. Show that \( \mathbb{F}^* \) is cyclic. (Hint : Let \( d \) be maximal so that there is an \( x \in \mathbb{F}^* \) with \( \text{ord}(x) = d \). Here \( \text{ord}(x) \) denotes the order of the element \( x \) in the group \( \mathbb{F}^* \). Consider the equation \( X^d - 1 \) in \( \mathbb{F} \).)

(b) (4 points) Consider the finite field \( \mathbb{F}_9 \) of order 9. Find all generators for the cyclic group \( \mathbb{F}_9^* \).

(c) (4 points) Use the first part to show that the field extension \( \mathbb{F}_{p^n}/\mathbb{F}_p \) is a separable extension. Here \( \mathbb{F}_{p^n} \) is the finite field with \( p^n \) elements.

9. (8 points) Let \( V \) be a finite dimensional vector space over \( \mathbb{Q} \). Consider a linear operator \( T : V \rightarrow V \) with characteristic polynomial \( c(x) \in \mathbb{Q}[x] \). Let \( f(x) \in \mathbb{Q}[x] \) be coprime to \( c(x) \). Show that the operator \( f(T) \) is invertible.

10. (7 points) Prove or disprove : Every finite group \( G \) is isomorphic to a subgroup of a dihedral group \( D_{2n} \) for some \( n \). Recall : that \( D_{2n} \) is the group of symmetries of a regular \( n \)-gon. (Remember to fully justify your answer.)

11. (5 points) Suppose that \( G \) is a finite set with an associative binary operation denoted by \( \circ \). Suppose that for all \( a, b \in G \) the equations

\[
 a \circ x = b \\
 x \circ a = b
\]

have at least one solution \( x \) in \( G \). Show that \( G \), with multiplication defined by \( \circ \), is a group.