1. Let \((A, \alpha), (B, \beta)\) and \((C, \gamma)\) be metric spaces and define
\[d((a_1, b_1), (a_2, b_2)) = \sqrt{\alpha(a_1, a_2)^2 + \beta(b_1, b_2)^2}.
\]
(a) Prove that \((A \times B, d)\) is a metric space. [2 marks]
(b) Suppose \(K\) is a compact subset of \(B\), \(V\) is an open subset of \(C\), and \(f\) is a continuous map from \((A \times B, d)\) to \((C, \gamma)\). Prove that
\[W = \{a \in A : f(a, b) \in V \text{ for all } b \in K\}\]
is an open subset of \(A\). [8 marks]

2. Solve the initial value problem,
\[f'''(x) - 3f''(x) + 4f(x) = 4x + 4, \quad f(0) = 2, \quad f'(0) = 4, \quad f''(0) = 8.
\]
[10 marks]

3. Let \(R = \{(u, v) \in \mathbb{R}^2 : u > 0, u^2v > 1\}\).
(a) Find the image of \(R\) under the map \((u, v) \mapsto (u^{-2}v^{-1}, u^{-1}v^{-2})\). [2 marks]
(b) Evaluate
\[
\int \int_{R} \frac{1}{u^4v^4 + u^2} \, du \, dv.
\]
Justify your answer. [8 marks]

4. Suppose \(f\) is a non-negative, decreasing function on \((0, \infty)\) such that \(\int_0^\infty f(x) \, dx < \infty\) and let \(g(x) = \sum_{n=1}^\infty f(2^n x)\) for each \(x > 0\). Prove that \(g(x) < \infty\) for all \(x > 0\), that the sum converges uniformly to \(g\) on any compact subinterval of \((0, \infty)\), and that \(\int_0^\infty g(x) \, dx = \int_0^\infty f(x) \, dx\). [10 marks]

5. Evaluate \(\int_{-\infty}^{+\infty} \frac{x^2}{1 + x^4} \, dx\). [10 marks]
6. Find three different Laurent series for \( f(z) = \frac{1}{z} + \frac{1}{(z + 1)^2} + \frac{1}{z - 2} \) about \( z_0 = 0 \) and state their regions of convergence.

7. Let \( \gamma \) be the limaçon \( r = \frac{3}{2} + 3 \cos \theta \) traversed one time counterclockwise. Evaluate the integrals

\[
\begin{align*}
\text{(a)} & \quad \int_{\gamma} \frac{e^{2z}}{(1 + z^2)^2} \, dz \\
\text{(b)} & \quad \int_{\gamma} \frac{\sin(\pi z)}{z^2 - 3z + 2} \, dz.
\end{align*}
\]

8. Let \( \Omega \) be a non-empty open subset of \( \mathbb{C} \), let \( (f_n)_{n=1}^{\infty} \) be a sequence of functions holomorphic on \( \Omega \), and let \( f : \Omega \to \mathbb{C} \) be a non-constant function. Suppose that \( f_n \to f \ (n \to \infty) \) uniformly on every compact subset of \( \Omega \). Prove that, if \( p \in \Omega \) and \( f(p) = 0 \), then for every open neighbourhood \( U \) of \( p \) in \( \Omega \) there exists \( N \in \mathbb{N} \) such that \( f_n \) has a zero in \( U \) for all \( n \geq N \).

9. **BONUS** Prove that there is no function \( f \) analytic in the disc \( D = \{ z \in \mathbb{C} : |z| < 2012 \} \) and such that \( |f(z)| \to \infty \) as \( |z| \to 2012^- \).