THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENT OF MATHEMATICS

Ph. D. Comprehensive Exam (Analysis)

May 14, 2012 10:00 a.m. – 1:00 p.m.

Answer completely as many questions as you are able. More credit will be given for several complete solutions than for many partial solutions.

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1. Let (A, α), (B, β) and (C, γ) be metric spaces and define

$$d((a_1, b_1), (a_2, b_2)) = \sqrt{\alpha(a_1, a_2)^2 + \beta(b_1, b_2)^2}\,.$$

[2 marks] (a) Prove that $(A \times B, d)$ is a metric space.

[8 marks] (b) Suppose K is a compact subset of B, V is an open subset of C, and f is a continuous map from $(A \times B, d)$ to (C, γ). Prove that

$$W = \{a \in A : f(a, b) \in V \text{ for all } b \in K\}$$

is an open subset of A.

[10 marks] **2.** Solve the initial value problem,

$$f'''(x) - 3f''(x) + 4f(x) = 4x + 4, \quad f(0) = 2,\ f'(0) = 4,\ f''(0) = 8\,.$$

3. Let $R = \{(u, v) \in \mathbb{R}^2 : u > 0, u^2 v > 1\}$.

[2 marks] (a) Find the image of R under the map $(u, v) \mapsto (u^{-2}v^{-1}, u^{-1}v^{-2})$.

[8 marks] (b) Evaluate

$$\iint_R \frac{1}{u^4 v^4 + u^2}\, du\, dv\,.$$

Justify your answer.

[10 marks] **4.** Suppose f is a non-negative, decreasing function on $(0, \infty)$ such that $\int_0^\infty f(x)\, dx < \infty$ and let $g(x) = \sum_{n=1}^\infty f(2^n x)$ for each $x > 0$. Prove that $g(x) < \infty$ for all $x > 0$, that the sum converges uniformly to g on any compact subinterval of $(0, \infty)$, and that $\int_0^\infty g(x)\, dx = \int_0^\infty f(x)\, dx$.

[10 marks] **5.** Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{1 + x^4}\, dx\,.$

[10 marks] **6.** Find three different Laurent series for $f(z) = \dfrac{1}{z} + \dfrac{1}{(z+1)^2} + \dfrac{1}{z-2}$ about $z_0 = 0$ and state their regions of convergence.

7. Let γ be the limaçon $r = \frac{3}{2} + 3\cos\theta$ traversed one time counterclockwise. Evaluate the integrals

[5 marks] (a) $\displaystyle\int_\gamma \frac{e^{2z}}{(1+z^2)^2}\,dz$

[5 marks] (b) $\displaystyle\int_\gamma \frac{\sin(\pi z)}{z^2 - 3z + 2}\,dz\,.$

[10 marks] **8.** Let Ω be a non-empty open subset of $\mathbb{C}$, let $(f_n)_{n=1}^{\infty}$ be a sequence of functions holomorphic on Ω, and let $f : \Omega \to \mathbb{C}$ be a non-constant function. Suppose that $f_n \longrightarrow f$ $(n \to \infty)$ uniformly on every compact subset of Ω. Prove that, if $p \in \Omega$ and $f(p) = 0$, then for every open neighbourhood U of p in Ω there exists $N \in \mathbb{N}$ such that f_n has a zero in U for all $n \geq N$.

[10 marks] **9. BONUS** Prove that there is no function f analytic in the disc $D = \{z \in \mathbb{C} : |z| < 2012\}$ and such that $|f(z)| \to \infty$ as $|z| \to 2012^-$.