THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination
Algebra

September 2013 3 hours

Instructions: Completely answer as many questions as you are able. More credit will be given for a complete solution than for several partial solutions. **Explain all answers fully.** Solutions will be graded based on correctness, completeness and clarity.

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1. (a) Let A and B be 5×5 matrices over $\mathbb{C}$ with the same minimal polynomial and characteristic polynomial, and with at least three distinct eigenvalues. Prove that A and B are similar.

   (b) Find an example of two 5×5 matrices A and B over $\mathbb{C}$ which are not similar but which have the same minimal polynomial and characteristic polynomial and two distinct eigenvalues.

2. Let T be a linear operator on a finite-dimensional vector space V over a field k. Prove that there exists a decomposition $V = X \oplus Y$ with X and Y T-invariant, such that $T|_X : X \to X$ is invertible and $T|_Y : Y \to Y$ is nilpotent.

3. (a) Define the **trace** $\mathrm{tr}(A)$ of an $n \times n$ matrix A, and prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $n \times n$ matrices A and B.

   (b) For an $n \times n$ matrix A over $\mathbb{C}$, show that $\mathrm{tr}(A)$ is equal to the sum of the eigenvalues of A (repeated according to multiplicity).

   (c) Show that if A is an $n \times n$ matrix and $\mathrm{tr}(AX) = 0$ for all $n \times n$ matrices X, then $A = 0$.

4. (a) State the classification of finite abelian groups.

   (b) List all abelian groups of order $16 \cdot 9 = 144$.

5. Let H be a subgroup of a group G with normalizer $N_G(H) = \{x \in G \mid x^{-1}Hx = H\}$ in G.

   (a) Prove that $|\{x^{-1}Hx \mid x \in G\}| = [G : N_G(H)]$, assuming that $N_G(H)$ has finite index in G.

   (b) Prove that if H has finite index in G, then H contains a subgroup M which is of finite index and normal in G.

   (c) Prove that if H is a proper subgroup of a finite group G, then $\cup_{x\in G}\, x^{-1}Hx$ is not the whole of G.

6. Recall that $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is the group of 2×2 matrices of determinant 1, which have entries in $\mathbb{Z}/p\mathbb{Z}$.

   (a) Show that the order of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is $p(p-1)(p+1)$.

   (b) Determine the number of 5-Sylow subgroups of $\mathrm{SL}_2(\mathbb{Z}/5\mathbb{Z})$.

7. (a) Define what it means for a complex number to be an **algebraic integer**.

   (b) If y is an algebraic integer, show that for some n there exists an $n \times n$ matrix A with entries in $\mathbb{Z}$ such that $AY = yY$, where $Y = [1, y, y^2, \ldots, y^{n-1}]^t$.

   (c) Prove that y is an algebraic integer if and only if it is an eigenvalue of a square matrix with entries in $\mathbb{Z}$.

8. Let R be a commutative ring with unity that is not a field.

   (a) Prove that the following conditions are equivalent.
   (i) The sum of two non-units is a non-unit.
   (ii) The non-unit elements form a proper ideal.
   (iii) The ring possesses a unique maximal ideal.

   (b) Show that $R = k[[x]]$, where k is a field, is an example of such a ring.

9. Let R be a nontrivial commutative ring with unity, and let M be a free R-module with finite basis $X = \{x_1, \ldots, x_m\}$.

   (a) Prove that every basis of M is finite.

   (b) Use Zorn's Lemma to show that R has a maximal ideal J.

   (c) Prove that every basis of M has m elements.

10. Let $F = \mathbb{Q}(\sqrt[4]{2})$ and $K = \mathbb{Q}(\sqrt[4]{2}, i)$.

    (a) Show that the extension of K over $\mathbb{Q}$ is Galois and compute its Galois group G. Explain fully.

    (b) Describe the subgroup H of G corresponding to F.

    (c) Deduce from part (b) that there is one and only one intermediate field between F and $\mathbb{Q}$.