THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 2, 2013 *3 hours*

Instructions: Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

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1. The Bernoulli numbers $B_0, B_1, \ldots$ are defined by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$$

(The function on the left hand side is defined to be equal to 1 at $t = 0$.)
a) What is the radius of convergence of the above Taylor series? Explain.
b) Compute B_0 and B_1 and show that $B_{2n+1} = 0$ for all $n \geq 1$.

2. a) Show that the series

$$\sum_{n=0}^{\infty} \left(\frac{1}{s+n} + \frac{1}{s-n}\right)$$

is pointwise convergent for all $s \in \mathbb{R} \backslash \mathbb{Z}$.
b) Use Weierstrass M-test (or any other method) to show that the resulting function is continuous on its domain $\mathbb{R} \backslash \mathbb{Z}$.

3. a) A function $f : X \to X$ from a metric space (X, d) to itself is called a *contraction* if there is a constant $K < 1$ such that $d(f(x), f(y)) \leq K d(x, y)$ for all $x, y \in X$.
a) Show that a contractive map from a *complete* metric space to itself has a unique fixed point. (x is a fixed point if $f(x) = x$).
b) Give counterexamples to show that both conditions (completeness of X and contractive property of f are needed in general).

4. Consider the *derivative operator* $T : C^1[0, 1] \to C[0, 1]$, $T(f) = f'$, from the space of continuously differentiable functions on the interval $[0, 1]$ to the space of continuous functions. Show that this map is not continuous with respect to the uniform metric $d(f, g) = \sup\{|f(x) - g(x)|;\ x \in [0,\ 1]\}$.

5. Show that the limit

$$\lim_{x\to\infty}\int_0^x \sin t^2 dt$$

exists. (Hint: use the substitution $u = t^2$ as a first step.)

6. a) Define what it means for a topological space to be *connected*, or *path connected*.
b) Show that a path connected topological space is connected.
c) Give an example of a topological space which is connected but not path connected.

7. Suppose $f(z)$ is an entire function and $|f(z)| \geq |e^z|$ for all z. Prove: $f(z) = ce^z$ for some constant c.

8. Let $D = \{z \in \mathbb{C}|\ |z| < 1\}$. Suppose $f : D \to \mathbb{C}$ is analytic and

$$|f(z)| \leq \frac{1}{1-|z|}$$

Show:

$$|f'(z)| \leq \frac{4}{(1-|z|)^2}$$

9. Evaluate

$$\frac{1}{2\pi i}\int_{|z|=1}\frac{z^2}{4e^z - z}dz$$

10. Let $D = \{z \in \mathbb{C}|\ |z| < 1\}$. Suppose $f : D \to D$ is analytic and has (at least) two fixed points (i.e. there are $a, b \in D$ such that $f(a) = a$, $f(b) = b$, $a \neq b$). Prove: $f(z) = z$.