THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 2, 2013 3 hours

Instructions: Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

1. The Bernoulli numbers \( B_0, B_1, \ldots \) are defined by

\[
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}
\]

(The function on the left hand side is defined to be equal to 1 at \( t = 0 \).)

a) What is the radius of convergence of the above Taylor series? Explain.

b) Compute \( B_0 \) and \( B_1 \) and show that \( B_{2n+1} = 0 \) for all \( n \geq 1 \).

2. a) Show that the series

\[
\sum_{n=0}^{\infty} \left( \frac{1}{s + n} + \frac{1}{s - n} \right)
\]

is pointwise convergent for all \( s \in \mathbb{R} \setminus \mathbb{Z} \).

b) Use Weierstrass \( M \)-test (or any other method) to show that the resulting function is continuous on its domain \( \mathbb{R} \setminus \mathbb{Z} \).

3. a) A function \( f : X \to X \) from a metric space \( (X, d) \) to itself is called a contraction if there is a constant \( K < 1 \) such that \( d(f(x), f(y)) \leq K d(x, y) \) for all \( x, y \in X \).

a) Show that a contractive map from a complete metric space to itself has a unique fixed point. (\( x \) is a fixed point if \( f(x) = x \).)

b) Give counterexamples to show that both conditions (completeness of \( X \) and contractive property of \( f \) are needed in general).

4. Consider the derivative operator \( T : C^1[0, 1] \to C[0, 1] \), \( T(f) = f' \), from the space of continuously differentiable functions on the interval \( [0, 1] \) to the space of continuous functions. Show that this map is not continuous with respect to the uniform metric \( d(f, g) = \sup \{|f(x) - g(x)|; \ x \in [0, 1]\} \).
5. Show that the limit
\[ \lim_{x \to \infty} \int_0^x \sin t^2 \, dt \]
exists. (Hint: use the substitution \( u = t^2 \) as a first step.)

6. a) Define what it means for a topological space to be connected, or path connected.
   b) Show that a path connected topological space is connected.
   c) Give an example of a topological space which is connected but not path connected.

7. Suppose \( f(z) \) is an entire function and \( |f(z)| \geq |e^z| \) for all \( z \). Prove: \( f(z) = ce^z \) for some constant \( c \).

8. Let \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Suppose \( f: D \to \mathbb{C} \) is analytic and
   \[ |f(z)| \leq \frac{1}{1 - |z|} \]
   Show:
   \[ |f'(z)| \leq \frac{4}{(1 - |z|)^2} \]

9. Evaluate
   \[ \frac{1}{2\pi i} \int_{|z|=1} \frac{z^2}{4e^z - z} \, dz \]

10. Let \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \). Suppose \( f: D \to D \) is analytic and has (at least) two fixed points (i.e. there are \( a, b \in D \) such that \( f(a) = a, f(b) = b, a \neq b \)). Prove: \( f(z) = z \).