UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

May 21, 2013 3 hours

Instructions: Answer completely as many questions as you can. More credit may be given for a complete solution than for several partial solutions.

(1) (a) State the three Sylow Theorems.
(b) Use these results to prove that any group of order 65 is cyclic.

(2) Let G be a finite group of order p^n acting on a finite set X. Prove that

$$|X| \equiv |X^G| \mathrm{mod}(p),$$

where $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$.

(3) (a) Define what it means for a group G to be *solvable*.
(b) Prove that a group G of invertible, upper-triangular matrices over the field k is solvable.

(4) Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{Z}[X]$. Prove that $\mathfrak{m}$ is not a principal ideal of $\mathbb{Z}[X]$.

(5) Consider the ring $\mathbb{Z}[X]$ of polynomials over $\mathbb{Z}$.
(a) Define what it means for $f \in \mathbb{Z}[X]$ to be *primitive*.
(b) Prove that it $f \in \mathbb{Z}[X]$ and $g \in \mathbb{Z}[X]$ are both primitive then $fg \in \mathbb{Z}[X]$ is also primitive.

(6) Let $n > 1$ and let $\mathbb{Z}_n$ be the ring of integer modulo n.
(a) Identify the units $\mathbb{Z}_n^*$ of $\mathbb{Z}_n$.
(b) Find a formula for the cardinality $|\mathbb{Z}_n^*|$, of $\mathbb{Z}_n^*$, in terms of n.
(c) Does there exist an n such that $|\mathbb{Z}_n^*| = 14$? Why or why not?

(7) Let V be the real vector space of functions $f : \mathbb{R} \to \mathbb{R}$ spanned by $\beta_0 = \{e^x, e^{-x}, xe^x, xe^{-x}\}$. Let $T : V \to V$ be the linear mapping given by $T(f) = f'$. Find the Jordan canonical form J of T and a basis β of V such that the matrix of T with respect to β is J.

(8) Let $P_n(\mathbb{R})$ be the real vector space of polynomials of degree at most n. Let $T : P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be defined by $T(f(x)) = f(-x)$. Give

$P_n(\mathbb{R})$ the inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt$$

(a) Find the minimal polynomial of T, the eigenvalues of T and a description of each eigenspace.
(b) Prove carefully that T is self-adjoint. Is T orthogonally diagonalisable, normal, orthogonal? Justify each answer.

(9) Let $\theta_7 \in \mathbb{C}$ be a primitive 7-th root of unity. What is the minimal polynomial of $\theta_7 + \theta_7^{-1}$ over $\mathbb{Q}$? Justify your answer.

(10) Prove that the centre of the ring of n by n matrices over a field F is $\{aI_n : a \in F\}$ where I_n is the n by n identity.

(11) Let $f(x) = x^4 - 2x^2 - 2 \in \mathbb{Q}[X]$,
(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$.
(b) Find the splitting field L of f over $\mathbb{Q}$ and its degree over $\mathbb{Q}$.
(c) Find generators and relations for the Galois group of $L/\mathbb{Q}$.

(12) Let $A \in M_{nn}(\mathbb{C})$ have rank 1. Show that $\det(A + I) = \mathrm{tr}(A) + 1$.