PhD Comprehensive Exam Part I: Analysis

1. Let $\mathcal{L} \subset \mathbb{C}$ be a real line passing through the origin. Prove that if points z_k, $k = 1, \ldots, n$ lie on one side of $\mathcal{L}$, then
$$\sum_{k=1}^{n} z_k \neq 0.$$

2. Suppose f and g are continuous functions on $[0, 1]$. Prove that
$$g(x) = (1-x)\left(g(0) - \int_0^x tf(t)\,dt\right) + x\left(g(1) - \int_x^1 (1-t)f(t)\,dt\right)$$
for all $x \in [0, 1]$ if and only if g is twice differentiable and $g'' = f$ on $[0, 1]$.

3. Let $S = \{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 \leq \sqrt{y^2 + z^2}\}$. Evaluate
$$\iiiint_S \left(w^2 + x^2 + y^2 + z^2\right) dw\,dx\,dy\,dz.$$

4. Suppose $f(x)$ is a function holomorphic on the open unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and continuous on the closed disc $\overline{\Delta}$. Suppose that $\operatorname{Im} f(z) = 0$ for $|z| = 1$. Give the best possible description of $f(x)$.

5. Let $P \subset \mathbb{R}^2$ be a polygon with non-self-intersecting boundary, having vertices $v_j = (x_j, y_j)$, for $j = 1, \ldots n$, and edges $\overline{v_1 v_2}, \ldots, \overline{v_{n-1} v_n}$ and $\overline{v_n v_1}$. Use Green's theorem to prove that the area of P is
$$\tfrac{1}{2}\left|(x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\right|.$$

6. Find the general form of a linear-fractional transformation that preserves two opposite points on the Riemann sphere.

7. Suppose that $f(z)$ is a holomorphic function with an isolated singularity at a point z_0. Suppose that the singularity is not removable. Prove that the function $e^{f(z)}$ has an essential singularity at z_0.

8. Let $C(\mathbb{R}^m, \mathbb{R})$ be the set of all continuous functions from $\mathbb{R}^m$ to $\mathbb{R}$ and define,
$$d(f, g) = \sum_{k=1}^{\infty} 2^{-k} \min(1, \max_{|x| \leq k} |f(x) - g(x)|).$$

   a) Prove that d is a metric on $C(\mathbb{R}^m, \mathbb{R})$.

   b) Suppose $f, f_n \in C(\mathbb{R}^m, \mathbb{R})$ for $n = 1, 2, \ldots$. Prove that $d(f_n, f) \to 0$ if and only if for every compact $K \subseteq \mathbb{R}^m$ and every $\varepsilon > 0$ there exists an N such that $|f_n(x) - f(x)| < \varepsilon$ whenever $x \in K$ and $n \geq N$.