Algebra Comprehensive exam
September 2014
Department of Mathematics

Instructions:

1. There are 12 questions.
2. Only 10 questions will be marked. Clearly indicate which 10 you would like considered.
3. No books or notes may be used.
4. Unless otherwise indicated, give complete justifications for your solutions. Answers without appropriate reasoning will not receive credit. There will be little or no partial credit – aim for complete solutions.

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1. Does there exist a finite abelian group of order 16 with 4 elements of order 4? If such a group M exists, we know that M is isomorphic to a group of the form

$$\mathbb{Z}/m_1\mathbb{Z} \oplus \mathbb{Z}/m_2\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/m_k\mathbb{Z},$$

where $m_1 \geq m_2 \geq \cdots \geq m_k$. What are k and m_i, and are they unique?

2. Show that every non-zero prime ideal in a principal ideal domain R is in fact a maximal ideal.

3. Let p be a prime number. Let $K/\mathbb{F}_p$ be a finite Galois extension. The purpose of this problem is to show that $\mathrm{Gal}(K/\mathbb{F}_p)$ is cyclic.
   (a) Show that the function $F : K \to K$ given by $F(x) = x^p$ is in fact a homomorphism.
   (b) Show that F is an automorphism fixing $\mathbb{F}_p$.
   (c) Suppose that the order of F is n in $\mathrm{Gal}(K/\mathbb{F}_p)$. Show that the polynomial $X^{p^n} - X$ vanishes on K.
   (d) Conclude that $\mathrm{Gal}(K/\mathbb{F}_p)$ is cyclic of order n.

4. Consider the $n \times n$ real matrix

$$B = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & n-1 \end{pmatrix}.$$

In other words B is a diagonal matrix with eigenvalues $-1, 1, 2, \ldots, n-1$. Show that there is no real matrix A with $A^2 = B$.

5. Show that the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ is Galois of degree 4. Compute its Galois group, and make sure you completely justify your answer.

6. Let N be a $n \times n$ complex matrix with $N^n = 0$. Show that $\det(I_n + N) = 1$.

7. Let p be a prime number.
   (a) What is a Sylow p-subgroup of a finite group G.
   (b) State, but do not prove, the Sylow theorems.
   (c) Find a Sylow 2-subgroup of S_4.

8. Let $f\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ be the homomorphism of abelian groups given by

$$f(v) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} v,$$

for $v \in \mathbb{Z}^2$. Describe $\ker(f)$ and $\operatorname{coker}(f)$ up to isomorphism as direct sums of cyclic groups. (Recall $\operatorname{coker}(f) := \mathbb{Z}^2/\operatorname{im}(f)$.)

9. (a) Prove that a $n \times n$ matrix over $\mathbb{C}$ satisfying $A^3 = I_3$ can be diagonalized.

   (b) Find a field k and a 3×3 matrix A satisfying $A^2 = I_3$ that cannot be diagonalized.

10. Show that any group of order 14 is isomorphic to either a cyclic or a dihedral group.

11. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4. Let L be the splitting field of f and suppose that $[L : \mathbb{Q}] = 8$. Prove or disprove: the group $G = \operatorname{Gal}(L/\mathbb{Q})$ is not abelian.

12. Suppose M and N are R-modules and I and J are ideals for which

    - $MI = 0$ and $JN = 0$;
    - $I + J = R$.

    Prove that $M \otimes_R N = 0$.