Department of Mathematics
Western University

PhD Comprehensive Examination Part I: Analysis

9:00-12:00, October 1, 2014

Instructions: *There are eight questions. Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.*

1. Show that

$$\int_{|z|=1} e^{\sin(1/z)}\, dz = 2\pi i.$$

2. Let (X, d) be a non-empty metric space such that for each $x \in X$ the closed unit ball centred at x, $B_x = \{y \in X : d(x, y) \leq 1\}$, is compact. Prove that (X, d) is complete.

3. Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^4}.$$

4. Show that every solution, $y = y(x)$, to the differential equation,

$$y''' + 2y'' - 2y = xe^x$$

is also a solution to the differential equation,

$$y^{(5)} - 3y''' + 4y' - 2y = 0.$$

5. Let

$$f(z) = \frac{1}{z} - \frac{1}{(z+1)^2}.$$

Find all possible Laurent expansions of f about $z_0 = 1$ and determine where each is valid.

6. For $(u, v) \in (0, \infty)^2$, define $F(u, v) = (v(1 + u)u, v(1 + u)/u)$. On large, clearly labeled axes sketch the region $F^{-1}((0, 1)^2)$ and clearly identify the curve $F^{-1}(\{(x, x) : 0 < x < 1\})$. Make the change of variable $(x, y) = F(u, v)$ to evaluate,

$$\int_0^1 \int_0^y \frac{dx\, dy}{y + \sqrt{xy}}.$$

7. Show that there exists no function f analytic in the unit disc $\{z : |z| < 1\}$ with the property that $|f(z)| \to \infty$ as $|z|$ increases to 1.

8. Suppose f_n, $n = 1, 2, \ldots$, are continuous functions from $[0, \infty)$ to $\mathbb{R}$ such that

$$\lim_{n \to \infty} \int_0^n |f_n(t)|\, dt = 0.$$

For $x \geq 0$, let

$$F_n(x) = \int_0^x f_n(t)\, dt, \quad n = 1, 2, \ldots.$$

Prove that F_n converges uniformly to 0 on $[0, m]$ for every $m > 0$.