UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

May 2014 3 hours

Instructions: Answer completely as many questions as you can. **All answers must be justified** unless stated otherwise. They will be graded according to correctness, completeness and clarity of presentation. More credit may be given for a complete solution than for several partial solutions.

(1) Find all finite abelian groups G with $|\mathrm{Aut}(G)|$ a prime number.

(2) Let p be a prime number, S_p the symmetric group on p letters, $\sigma \in S_p$ a p-cycle and $\tau \in S_p$ a transposition. Show that σ and τ generate S_p. Justify your answer carefully.

(3) Let F be a finite field of characteristic p, and G a subgroup of order p^a, $a \in \mathbb{N}$ of the group $GL(n, F)$. Show that there is a non-zero vector $\mathbf{v}$ of F^n such that $g\mathbf{v} = \mathbf{v}$ for all $g \in G$.

(4) Let $f(x) \in F[X]$ be an irreducible polynomial of degree d over a field F. Let K/F be a finite field extension of degree n. Show that if $\gcd(n, d) = 1$, then $f(x)$ is irreducible as a polynomial in $K[X]$.

(5) Determine the Galois group of $f(x) = x^5 - 4x + 2 \in \mathbb{Q}[x]$. Justify your answer.

Hint: You may use Question (2) here, even if you haven't solved it.

(6) Show that the identity map is the only field automorphism of the real numbers. Show that this is not true for the complex numbers.

Hint: Show that $a < b$ implies $\sigma(a) < \sigma(b)$ for any $a, b \in \mathbb{R}$ and any field automorphism σ of $\mathbb{R}$.

(7) Let V be an n-dimensional vector space over an algebraically closed field F, and let $T : V \to V$ be a linear map. Show that there exists a basis $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ for V such that the matrix of T with respect to B is upper triangular. Find a counterexample over a non-algebraically closed field.

(8) Let $A \subset M_{nn}(\mathbb{R})$ be a subspace of pairwise commuting symmetric matrices. Show that $\dim(A) \leq n$.

(9) Let V be a finite dimensional vector space over a field F. Find all (one-sided) zero-divisors in the ring $\mathrm{End}_F(V)$ of linear maps $V \to V$. Justify your answer.

(10) Let $R = \mathbb{Z}[T, T^{-1}]$ be the ring of Laurent polynomials in one variable.
(a) Show that the units in R are $R^\times = \{\pm T^n : n \in \mathbb{Z}\}$.
(b) Find all ring homomorphisms $f : R \to R$.

(11) Let A be a commutative ring (with identity element). Show that if A has finite cardinality, then every prime ideal of A is maximal.

(12) Let A be a commutative ring (with identity element) and let $I \triangleleft A$ be a nilpotent ideal of A. That is, there exists $k \in \mathbb{N}$ such that $I^k = 0$. Let $\pi : A \to A/I$ be the canonical projection. Show that $a \in A$ is invertible in A if and only if $\pi(a)$ is invertible in A/I.