The University of Western Ontario
Department of Mathematics
Ph.D. Comprehensive Examination (Analysis)

May 14 2014 3 hours

Instructions: Justify all your answers. More credit will be given for a complete solution than for several partial solutions. Each problem is worth 10 marks.

(1) (a) Show that for all $x \in \mathbb{R}$

$$f(x) := \left(\int_0^x e^{-t^2}\,dt\right)^2 + \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}\,dt = \frac{\pi}{4}$$

(b) Deduce that $\displaystyle\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$.

(2) Given $x \in \mathbb{R}$, $x > 0$, let $\langle x \rangle \in [0,1)$ be the fractional part of x. For $n \in \mathbb{N}$, define $f_n(x) = \langle nx \rangle$ and consider the series $f(x) = \displaystyle\sum_{n\geq 1} \frac{f_n(x)}{n^2}$.

(a) Show that f converges uniformly on $\mathbb{R}$.

(b) For a fixed n, find the discontinuities of the function $x \mapsto f_n(x)$ by computing the one-sided limits of the function.

(c) Show that f is continuous at any irrational number.

(d) Show that f is not continuous at any rational number.

(e) Show that f is Riemann integrable on any bounded interval.

(3) Let $X : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $(2xz^3 + 6y)\mathbf{i} + (6x - 2yz)\mathbf{j} + (3x^2z^2 - y^2)\mathbf{k}$. Compute the line integral $\displaystyle\int_C X \cdot d\mathbf{r}$ from $(1,-1,1)$ to $(2,1,-1)$ where C is the curve

$$C(t) = \big(2 - \cos(\pi t),\ 1 - 2\cos(\pi t),\ 1 - 8x^2\big)$$

(4) Let C be the set of continuous real-valued functions on $[0,1]$. Given $f, g \in C$, define

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)| \quad \text{and} \quad \rho(f,g) = \int_0^1 |f(t) - g(t)|\,dt$$

(a) Show that d and ρ are metrics on C.

(b) Prove that (C,d) is complete.

(c) Prove that the identity map $\mathrm{id} : (C,d) \to (C,\rho)$ is continuous.

(d) Is the identity map $\mathrm{id} : (C,\rho) \to (C,d)$ a homeomorphism? Explain.

(e) Show that (C,ρ) is not complete.

continued on the back...

(5) Compute the integral

$$\int_{|z|=1/2} \frac{e^{1/z}}{1-z}\, dz.$$

(6) Let f be a holomorphic function in a disc $U_R = \{|z| < R\}$, $f(0) = 0$, and $|f(z)| < M$ for all $z \in U_R$.

(a) Prove that $|f(z)| \leq \dfrac{M}{R}|z|$ in U_R, and that $|f'(0)| \leq \dfrac{M}{R}$.

(b) Show that the equality $|f'(0)| = \dfrac{M}{R}$ holds only if $f(z) = e^{i\alpha}\dfrac{M}{R}z$.

(7) Prove that if two functions $f_1(z)$ and $f_2(z)$ are holomorphic in a domain $D \subset \mathbb{C}$, and agree on a set E which has a point of accumulation $a \in D$, then $f_1 \equiv f_2$ in D.

(8) Let $G = \{z \in \mathbb{C} : |z| < 2, z \neq \pm 1\}$. Find all bijective conformal maps: $\phi : G \to G$.