Algebra Comprehensive Exam, October 1st 2015

Instructions: Answer completely as many questions as you can. More credit will be given for a complete, clearly written solution than for several partial solutions. Each question is of equal value.

1. Let A be an $n \times n$ matrix with n distinct complex eigenvalues, for an integer $n \geq 1$. Let $\mathrm{Mat}_{n\times n}$ be the vector space of $n \times n$ matrices over $\mathbb{C}$. Consider the linear operator $T_A : \mathrm{Mat}_{n\times n} \to \mathrm{Mat}_{n\times n}$ given by $T_A(X) = AX - XA$. What is $\dim \operatorname{image} T_A$? [Hint: what is $\ker T_A$?]

2. Find all abelian groups G, up to isomorphism, with the property that G has a subgroup $H \cong \mathbb{Z}/4\mathbb{Z}$ and for which $G/H \cong \mathbb{Z}/8\mathbb{Z}$.

3. (a) Show that the group of units in the ring $\mathbb{Z}/8\mathbb{Z}$ is not cyclic.
   (b) Show that, if p is prime, then the group of units in $\mathbb{Z}/p\mathbb{Z}$ is cyclic.

4. Let F be a field, and let $G = GL_2(F)$, the group of 2×2 matrices with entries in F. Suppose $A \in G$ is an element of finite order k, for some $k \geq 1$.
   (a) Suppose $F = \mathbb{C}$. Show that A is diagonalizable.
   (b) Suppose $F = \mathbb{R}$. Show that A need not be diagonalizable by giving a counterexample.
   (c) Suppose $F = \overline{\mathbb{F}}_2$, an algebraically closed field of characteristic 2. Must A be diagonalizable? Prove or disprove.

5. Suppose that a and b are relatively prime elements in a Unique Factorization Domain R. Show that there are no nonzero R-module homomorphisms $f\colon R/(a) \to R/(b)$.

6. Let p be a prime. Show that any group G of order p^2 is abelian.

7. Show that no group of order 30 is simple.

8. Show that the additive group $\mathbb{Q}$ is not isomorphic to the product of two non-trivial groups.

9. Let F be a subfield of $\mathbb{R}$, and let $f(X) \in F[X]$ be irreducible with a non-real root α of absolute value one. Show that $1/\beta$ is a root of $f(X)$ for every root $\beta \in \mathbb{C}$.

10. Let E/F be a field extension. Let $f(X) \in F[X]$ be irreducible and α_1, α_2, β_1, $\beta_2 \in E$ be roots of $f(X)$. Assume $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$.
    (a) Show that $F(\alpha_1)$ and $F(\alpha_2)$ are isomorphic extensions of F.
    (b) Are $F(\alpha_1, \alpha_2)$ and $F(\beta_1, \beta_2)$ always isomorphic extensions of F?

11. Let E be the splitting field of $f(X) = X^4 - 14X^2 + 9$ over $\mathbb{Q}$.
    1. Compute $\mathrm{Gal}(E/\mathbb{Q})$. (Hint: The roots of $f(X)$ are $\pm\sqrt{2} \pm \sqrt{5}$.)
    2. Verify that each subgroup of $\mathrm{Gal}(E/\mathbb{Q})$ is the Galois group $\mathrm{Gal}(E/L)$ of an intermediate field $\mathbb{Q} \subset L \subset E$.