Instructions: Answer completely as many questions as you can. More credit will be given for a complete, clearly written solution than for several partial solutions. Each question is of equal value.

1. Let $A$ be an $n \times n$ matrix with $n$ distinct complex eigenvalues, for an integer $n \geq 1$. Let $\text{Mat}_{n \times n}$ be the vector space of $n \times n$ matrices over $\mathbb{C}$. Consider the linear operator $T_A : \text{Mat}_{n \times n} \rightarrow \text{Mat}_{n \times n}$ given by $T_A(X) = AX -XA$. What is $\text{dim} \text{image} \ T_A$? [Hint: what is $\text{ker} \ T_A$?]

2. Find all abelian groups $G$, up to isomorphism, with the property that $G$ has a subgroup $H \cong \mathbb{Z}/4\mathbb{Z}$ and for which $G/H \cong \mathbb{Z}/8\mathbb{Z}$.

3. (a) Show that the group of units in the ring $\mathbb{Z}/8\mathbb{Z}$ is not cyclic.
   (b) Show that, if $p$ is prime, then the group of units in $\mathbb{Z}/p\mathbb{Z}$ is cyclic.

4. Let $F$ be a field, and let $G = GL_2(F)$, the group of $2 \times 2$ matrices with entries in $F$. Suppose $A \in G$ is an element of finite order $k$, for some $k \geq 1$.
   (a) Suppose $F = \mathbb{C}$. Show that $A$ is diagonalizable.
   (b) Suppose $F = \mathbb{R}$. Show that $A$ need not be diagonalizable by giving a counterexample.
   (c) Suppose $F = \mathbb{F}_2$, an algebraically closed field of characteristic 2. Must $A$ be diagonalizable? Prove or disprove.

5. Suppose that $a$ and $b$ are relatively prime elements in a Unique Factorization Domain $R$. Show that there are no nonzero $R$-module homomorphisms $f : R/(a) \rightarrow R/(b)$.

6. Let $p$ be a prime. Show that any group $G$ of order $p^2$ is abelian.

7. Show that no group of order 30 is simple.

8. Show that the additive group $\mathbb{Q}$ is not isomorphic to the product of two non-trivial groups.

9. Let $F$ be a subfield of $\mathbb{R}$, and let $f(X) \in F[X]$ be irreducible with a non-real root $\alpha$ of absolute value one. Show that $1/\beta$ is a root of $f(X)$ for every root $\beta \in \mathbb{C}$.

10. Let $E/F$ be a field extension. Let $f(X) \in F[X]$ be irreducible and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in E$ be roots of $f(X)$. Assume $\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$.
    (a) Show that $F(\alpha_1)$ and $F(\alpha_2)$ are isomorphic extensions of $F$.
    (b) Are $F(\alpha_1, \alpha_2)$ and $F(\beta_1, \beta_2)$ always isomorphic extensions of $F$?

11. Let $E$ be the splitting field of $f(X) = X^4 - 14X^2 + 9$ over $\mathbb{Q}$.
    1. Compute $\text{Gal}(E/\mathbb{Q})$. (Hint: The roots of $f(X)$ are $\pm \sqrt{2} \pm \sqrt{5}$.)
    2. Verify that each subgroup of $\text{Gal}(E/\mathbb{Q})$ is the Galois group $\text{Gal}(E/L)$ of an intermediate field $L \subset E$. 