THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 8, 2015 *3 hours*

Instructions: Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

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1. (a) Define Lipschitz continuous functions on the interval $I = [-1, 1]$.
   (b) Show that the uniform limit of Lipschitz continuous functions on the interval I may not be Lipschitz continuous.
   (c) Show that if a sequence $(f_m(x))_{m=1}^{\infty}$ converges uniformly on I, and all $f_m(x)$ are Lipschitz continuous with a uniform constant K, then the limit is also Lipschitz continuous with the constant K.

2. Let $f(x)$ be a function defined on $(0, 1]$ such that f is Riemann integrable on $[c, 1]$ for any $0 < c < 1$. Define
$$\int_0^1 f(x)\,dx = \lim_{c\to 0^+} \int_c^1 f(x)\,dx.$$
   (a) Show that if $f(x)$ is Riemann integrable on $[0, 1]$, then the standard definition of $\int_0^1 f(x)\,dx$ using Riemann sums and the definition given above agree.
   (b) Give example of a function which is not Riemann integrable on $[0, 1]$, but the limit above exists.

3. Consider the function
$$F(x) = \begin{cases} \dfrac{xy^3}{x^2 + y^4}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$
   (a) Show that $F(x, y)$ is continuous at the origin.
   (b) Define what it means for a function $f(x, y)$ to be differentiable at $(0, 0)$.
   (c) Show that $F(x, y)$ above is not differentiable at the origin.

4. Let $I = (-1, 1)$ be the open subinterval of $\mathbb{R}$. Let $\mathcal{S}$ be the space of bounded continuous functions on I. Define

$$\rho(f, g) = \sup_{x \in I} |f(x) - g(x)|\,, \quad f, g \in \mathcal{S}.$$

   (a) Prove that ρ is a metric on $\mathcal{S}$ and that $(\mathcal{S}, \rho)$ is a complete metric space.
   (b) Prove that the map $H : (\mathcal{S}, \rho) \to \mathbb{R}$, given by $H(f) = f(0)$, is continuous.

5. Is there a polynomial $P(z)$ such that $P(z) \cdot e^{1/z}$ is an entire function? Justify your answer (i.e., give an example or prove it does not exist).

6. Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4}\, dx\,.$$

7. Let f be a non-constant entire function. Show that the image of f is dense in $\mathbb{C}$.

8. Let $(z_n)_{n=1}^{\infty}$ be a sequence of distinct complex numbers such that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|z_n|^3}$ converges, and let

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{(z - z_n)^2} - \frac{1}{z_n^2} \right)\,.$$

   Prove that f is meromorphic on $\mathbb{C}$ and find all its poles.