UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

May 22, 2015 3 hours

Instructions: Answer completely as many questions as you can. More credit may be given for a complete solution than for several partial solutions.

(1) Let \((V, <, >)\) be an inner product space over \(\mathbb{C}\), and let \(T : V \rightarrow V\) be a linear operator.

(a) Define the adjoint \(T^* : V \rightarrow V\) (just say how it is defined, not why it exists).

(b) Suppose that \(W \subseteq V\) is a \(T\)-invariant subspace. Show that \(W^\perp\) is \(T^*\)-invariant.

(c) Show that if \(\lambda\) is an eigenvalue of \(T\) then \(\overline{\lambda}\) is an eigenvalue of \(T^*\).

(2) Let \(A = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{pmatrix}\).

(a) Find the characteristic polynomial of \(A\).

(b) Find \(E_\lambda\) (the eigenspace of \(\lambda\)) for each eigenvalue \(\lambda\) of \(A\).

(c) Find the Jordan canonical form of \(A\).

(3) Let \(A = \begin{pmatrix} 4 & 3 & 5 \\ 2 & 4 & 2 \\ 3 & 1 & 3 \end{pmatrix}\). Find \(d_1, d_2, d_3 \in \mathbb{Z}\) such that \(d_1|d_2|d_3\) and \(A\) is equivalent to \(D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}\).

(4) Examples and counter-examples.

(a) Give an example of a \(U.F.D.\) that is not a \(P.I.D.\).

(b) Give an example of a linear transformation \(\alpha : V \rightarrow V\) over a field \(k\) such that \(V\) is cyclic as a \(k[x]\)-module, but decomposable as a \(k[x]\)-module.
(c) Let $a = 3 + 4i, b = 1 + 2i \in \mathbb{Z}[i]$. Write

$$a = bq + r$$

where $r, q \in \mathbb{Z}[i]$, and $N(r) < N(b) = 5$. (Here $N(a + bi) = a^2 + b^2$ is the complex norm.)

(5) Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ and let $f : V \to V$ be a linear transformation. Prove that there exists a basis $B = \{v_1, ..., v_n\}$ of $V$ such that $f$ is in upper-triangular form with respect to $B$.

(6) (a) Let $A$ be a commutative ring with $1 \in A$. Prove that $A$ has a maximal proper ideal $M$.

(b) Prove that the rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic over any field $F$.

(7) (a) Show that the polynomial $f(x) = x^4 - 5$ is irreducible over $\mathbb{Q}$.

(b) Find the splitting field $K$ of $f(x)$ over $\mathbb{Q}$.

(c) Find the Galois group of $K/F$.

(8) Let $p$ be a prime number and $\mathbb{F}_p$ be a field with $p$-elements. Let $GL_4(\mathbb{F}_p)$ be a group of invertible matrices over $\mathbb{F}_p$ of size 4 by 4, and let $U_4(\mathbb{F}_p)$ be an upper triangular subgroup of $GL_4(\mathbb{F}_p)$ with all diagonal elements equal to 1. Show that $U_4(\mathbb{F}_p)$ is a $p$-Sylow subgroup of $GL_4(\mathbb{F}_p)$.

(9) Let $p$ be a prime number and let $\mathbb{F}_p$ be a field with $p$-elements. Determine the number of quadratic monic irreducible polynomials over $\mathbb{F}_p$. (A monic polynomial means that its leading coefficient is 1.)

(10) Let $G$ be a group with 21 elements. Show that:

(a) $G$ has a unique Sylow subgroup $P$ of order 7.

(b) $P$ is a normal subgroup of $G$ and that there exists an element $\sigma \in G$ such that $\sigma \neq 1$ and $\sigma^3 = 1$.

(c) Assume that $G$ as above, is not cyclic. Show that $G$ is a semi-direct product $G = P \rtimes \{1, \sigma, \sigma^2\}$ where $P = \{1, y, \ldots, y^6\}$ and $\sigma \tau \sigma^{-1} = y^2$, or $\sigma \tau \sigma^{-1} = y^4$.

(d) Show that both groups $G$ described in (c), are isomorphic.

(11) (a) Show that if group $G$ we have $\sigma^2 = 1$ for all $\sigma \in G$, then $G$ is abelian.

(b) Let $p$ be an odd prime number and let $\mathbb{F}_p$ be a field with $p$-elements. Consider the group $G = U_3(\mathbb{F}_p)$. This means that $G$ is a group of all $3 \times 3$ upper triangular invertible matrices over $\mathbb{F}_p$ with diagonal elements all equal to 1. Show that $\sigma^p = 1$ for all $\sigma \in G$, but $G$ is not an abelian group.
(12) Decide which of the following extensions of $\mathbb{Q}$ are Galois extensions of $\mathbb{Q}$, and explain your answer carefully.

(a) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$.

(b) $\mathbb{Q}(\sqrt{2}, \sqrt{-1})(\sqrt{1 + \sqrt{2}})/\mathbb{Q}$.

(c) $\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}$.

(d) $\mathbb{Q}(\sqrt{7})(\sqrt{1 + \sqrt{7}})/\mathbb{Q}$.