UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

May 22, 2015 3 hours

Instructions: Answer completely as many questions as you can. More credit may be given for a complete solution than for several partial solutions.

(1) Let $(V, <,>)$ be an inner product space over $\mathbb{C}$, and let $T : V \to V$ be a linear operator.
   (a) Define the *adjoint* $T^* : V \to V$ (just say how it is defined, not why it exists).
   (b) Suppose that $W \subseteq V$ is a T- invariant subspace. Show that $W^\perp$ is T^*-invariant.
   (c) Show that if λ is an eigenvalue of T then $\overline{\lambda}$ is an eigenvalue of T^*.

(2) Let $A = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{pmatrix}$.
   (a) Find the characteristic polynomial of A.
   (b) Find E_λ (the eigenspace of λ) for each eigenvalue λ of A.
   (c) Find the Jordan canonical form of A.

(3) Let $A = \begin{pmatrix} 4 & 3 & 5 \\ 2 & 4 & 2 \\ 3 & 1 & 3 \end{pmatrix}$. Find $d_1, d_2, d_3 \in \mathbb{Z}$ such that $d_1 | d_2 | d_3$ and A is equivalent to $D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}$.

(4) Examples and counter-examples.
   (a) Give an example of a *U.F.D.* that is not a *P.I.D.*
   (b) Give an example of a linear transformation $\alpha : V \to V$ over a field k such that V is cyclic as a $k[x]$-module, but decomposable as a $k[x]$-module.

(c) Let $a = 3 + 4i, b = 1 + 2i \in \mathbb{Z}[i]$. Write

$$a = bq + r$$

where $r, q \in \mathbb{Z}[i]$, and $N(r) < N(b) = 5$. (Here $N(a + bi) = a^2 + b^2$ is the complex norm.)

(5) Let V be an n-dimensional vector space over $\mathbb{C}$ and let $f : V \to V$ be a linear transformation. Prove that there exists a basis $B = \{v_1, ..., v_n\}$ of V such that f is in upper-triangular form with respect to B.

(6) (a) Let A be a commutative ring with $1 \in A$. Prove that A has a maximal proper ideal M.
(b) Prove that the rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic over any field F.

(7) (a) Show that the polynomial $f(x) = X^4 - 5$ is irreducible over $\mathbb{Q}$.
(b) Find the splitting field K of $f(x)$ over $\mathbb{Q}$.
(c) Find the Galois group of K/F.

(8) Let p be a prime number and $\mathbb{F}_p$ be a field with p-elements. Let $GL_4(\mathbb{F}_p)$ be a group of invertible matrices over $\mathbb{F}_p$ of size 4 by 4, and let $U_4(\mathbb{F}_p)$ be an upper triangular subgroup of $GL_4(\mathbb{F}_p)$ with all diagonal elements equal to 1. Show that $U_4(\mathbb{F}_p)$ is a p-Sylow subgroup of $GL_4(\mathbb{F}_p)$.

(9) Let p be a prime number and let $\mathbb{F}_p$ be a field with p-elements. Determine the number of quadratic monic irreducible polynomials over $\mathbb{F}_p$. (A *monic polynomial* means that its leading coefficient is 1.)

(10) Let G be a group with 21 elements. Show that:
(a) G has a unique Sylow subgroup P of order 7.
(b) P is a normal subgroup of G and that there exists an element $\sigma \in G$ such that $\sigma \neq 1$ and $\sigma^3 = 1$.
(c) Assume that G as above, is not cyclic. Show that G is a semi-direct product $G = P \rtimes \{1, \sigma, \sigma^2\}$ where $P = \{1, y, \ldots, y^6\}$ and $\sigma\tau\sigma^{-1} = y^2$, or $\sigma\tau\sigma^{-1} = y^4$.
(d) Show that both groups G described in (c), are isomorphic.

(11) (a) Show that if group G we have $\sigma^2 = 1$ for all $\sigma \in G$, then G is abelian.
(b) Let p be an odd prime number and let $\mathbb{F}_p$ be a field with p-elements. Consider the group $G = U_3(\mathbb{F}_p)$. This means that G is a group of all 3×3 upper triangular invertible matrices over $\mathbb{F}_p$ with diagonal elements all equal to 1. Show that $\sigma^p = 1$ for all $\sigma \in G$, but G is not an abelian group.

(12) Decide which of the following extensions of $\mathbb{Q}$ are Galois extensions of $\mathbb{Q}$, and explain your answer carefully.
(a) $\mathbb{Q}(\sqrt[4]{2})/\mathbb{Q}$.
(b) $\mathbb{Q}(\sqrt{2}, \sqrt{-1})(\sqrt{1+\sqrt{2}})/\mathbb{Q}$.
(c) $\mathbb{Q}(\sqrt{2}, \sqrt{-1})/\mathbb{Q}$.
(d) $\mathbb{Q}(\sqrt{7})(\sqrt{1+\sqrt{7}})/\mathbb{Q}$.