THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 15, 2015 *1:00-4:00 pm*

Instructions: Answer as many questions as you can. More credit will be given for several complete solutions than for many partial solutions.

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1. Find the number of roots of the polynomial $z^9 - 6z^4 + 3z - 1$ in $|z| < 1$.

2. Evaluate the integral:

$$\int_{|z|=4} \frac{e^{\frac{1}{z-1}}}{z-2}\, dz$$

3. Does there exist an entire function $f(z)$ such that

$$f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^{2015}}$$

for $n = 1, 2, 3, \ldots$?

4. For which $z \in \mathbb{C}$ does the series

$$\sum_{n=1}^{\infty} \Big(\frac{z^n}{(n+1)!} + \frac{n}{z^n} \Big)$$

converge ?

5. Let A be a positive definite real n by n matrix. Show that

$$\int_{\mathbb{R}^n} e^{-X^t A X} dX = \frac{\pi^{n/2}}{\sqrt{\det A}}$$

(You can use the fact that $\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$. Reduce the problem to 1-dimensional integrals. First work out the case where A is diagonal.)

6. Let $f : \mathbb{R} \to \mathbb{C}$ be a 2π-periodic *infinitely differentiable* function. For $n \in \mathbb{Z}$, let

$$\hat{f}(n) = \int_0^{2\pi} f(x)e^{-inx}dx$$

be the n-th *Fourier coefficient* of f. Show that $\hat{f}(n)$ is *rapid decay.* That is for any positive integer k,

$$|n^k \hat{f}(n)| \to 0, \quad \text{as } n \to \infty.$$

(Hint: you can use the Riemann-Lebesgue lemma: for any L^1-function g, $|\hat{g}(n)| \to 0$ as $|n| \to \infty$. Use integration by parts.)

7. State the *inverse function theorem.* Give an example of a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that for all (x, y), $Jf(x, y) \neq 0$, but f is not injective. (Jf denotes the Jacobian of f).

8. a) Use the *divergence theorem* to show that for any bounded domain $U \subset \mathbb{R}^2$ with smooth boundary ∂U and smooth functions φ, ψ on Ω, we have (Green's second identity)

$$\int_U (\psi\Delta\varphi - \varphi\Delta\psi)\, dxdy = \oint_{\partial U} \left(\psi\frac{\partial\varphi}{\partial\mathbf{n}} - \varphi\frac{\partial\psi}{\partial\mathbf{n}}\right) dl,$$

where $\frac{\partial\varphi}{\partial\mathbf{n}}$ is the directional derivative of φ in the direction of the outward pointing normal to the boundary, and $\Delta\varphi = \varphi_{xx} + \varphi_{yy}$ is the Laplacian of φ.

b) Use the above identity to show that for smooth function with compact support $f : \mathbb{R}^2 \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^2} (\ln r)\, \Delta f = -2\pi f(0),$$

where $r = \sqrt{x^2 + y^2}$.