THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 4, 2016 *3 hours*

Instructions: Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

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1. Show that for any smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ with *compact support* we have

$$\int_{\mathbb{R}^3} \frac{\Delta f(x)}{|x|} d^3x = 4\pi f(0),$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator. (Hint: Try to use *Green's second identity*: for any compact domain U with smooth boundary ∂U and smooth functions f, g on U, we have

$$\int_U (f\Delta g - g\Delta f) dv = \int_{\partial U} (f\frac{\partial g}{\partial \mathbf{n}} - g\frac{\partial f}{\partial \mathbf{n}}) ds,$$

where $\frac{\partial f}{\partial \mathbf{n}} = \nabla f \cdot \mathbf{n}$ is the directional derivative of f along the unit normal vector to the boundary of U.)

2. Let $u : [-1, 1] \to \mathbb{R}$ be a smooth function such that $u(1) = u(-1) = 0$. Show that

$$\int_{-1}^{+1} u^2(s) ds \leq 4 \int_{-1}^{+1} (u'(s))^2 ds$$

(Hint: Using the fundamental theorem of calculus, write $u(s) = \int_{-1}^{s} u'(t)dt$. Then try to use the Cauchy-Schwartz inequality $\int (fg) \leq (\int f^2)^{1/2} (\int g^2)^{1/2}$ to estimate the latter integral.)

3. Let $\theta \in \mathbb{R}$ be an *irrational number.*
1) Show that the set of numbers $n\theta \, (mod\, 1), n = 1, 2, 3, \ldots$ is dense in the unit interval $[0, 1]$.
2) Show that for any periodic function of the form $f(x) = \sum_k a_k e^{2\pi ikx}$, where the sum is over a *finite set of integers*, we have

$$\lim_{n\to\infty} \frac{1}{n+1} (\sum_{m=0}^{n} f(m\theta)) = \int_0^1 f(x)dx$$

4. For $n = 1, 2, \ldots,$ let

$$\gamma_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n}) - \ln n.$$

Show that $\lim_{n\to\infty} \gamma_n$ exists. The limit is known as Euler's constant γ. (Hint: show that the sequence is positive and decreasing).

5. How many zeros does the polynomial $p(z) = z^7 + z^4 + 5z^3 + 1$ have in the annulus $1 < |z| < 2$?

6. Evaluate the integral:

$$\int_0^{2\pi} \frac{d\varphi}{1 - e^{\frac{i\pi}{7}} \cos\varphi + \frac{1}{4} e^{i\frac{2\pi}{7}}}.$$

7. Prove: if $f(z)$ is an entire function such that $|f(z)||Im(z)|^2 \leq 1$ then $f(z) \equiv 0$.

8. Suppose $U \subset \mathbb{C}$ is an open set, $z_0 \in U$, $f : U - \{z_0\} \to \mathbb{C}$ is a continuous function, and $\int_\Gamma f(z)dz = 0$ whenever Γ is the boundary of a closed rectangle in $U - \{z_0\}$. Either
prove that there is a holomorphic function $g : U \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in U - \{z_0\}$

or

state that this is not true and provide a counterexample.