UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

September 2016 3 hours

Instructions: Answer completely as many questions as you can. More credit may be given for a complete solution than for several partial solutions. Make sure that you justify your answers carefully. The credit an answer receives depends both on mathematical correctness and on quality of exposition.

(1) (a) List all abelian groups (up to isomorphism) of order 200 that have no elements of order 40.
(b) Explain why the number of isomorphism classes of abelian groups of order p^n is independent of the prime p.

(2) Let p be a prime number. Assume that G is a finite group such that every element of G has order p^n for some $n \geq 0$. Prove that G has order p^N for some $N \geq 0$. State clearly which theorems (from finite group theory) you use in your proof.

(3) (a) Find the splitting field K of the given polynomial f over k in each case. In each case express your answer in the form $k(x_1, \ldots, x_n)$ for appropriate complex numbers $\{x_i\}$.
(i) $f(x) = x^3 - 3$ over $k = \mathbb{Q}$.
(ii) $f(x) = x^2 + 3$ over $k = \mathbb{R}$.
(iii) $f(x) = x^n + 1$ over $k = \mathbb{Q}$.
(b) Find the degree of each splitting field in (i), (ii) and (iii) and identify the Galois group $G = \mathrm{Gal}(K/k)$ in each case.

(4) Let $k \subseteq K$ be a finite extension of fields.
(a) Define what it means for $k \subseteq K$ to be *separable*.
(b) Define what it means for $k \subseteq K$ to be *normal*.
(c) Find a finite extension L of $\mathbb{Q}$ that is not normal.
(d) Does there exist a finite extension field of $\mathbb{Q}$ that is not separable? Why or why not?

(5) Let $f(x) = x^3 - 2x^2 + 3x - 5$. Assume that f has roots α, β and γ. Calculate $\alpha^2 + \beta^2 + \gamma^2$.

(6) Let A be an $n \times m$-matrix with entries in some field F.
   (a) Define what a reduced row-echelon form of A is.
   (b) Show that the reduced row-echelon form of A is unique.
   Hint: Assume that there are two distinct reduced row-echelon forms and look at the first column where they differ.

(7) Let $A \in \mathbb{C}^{n \times n}$. Show that the Jordan canonical form of A has exactly one Jordan block per eigenvalue if and only if there is no non-zero polynomial $p \in \mathbb{C}[x]$ of degree less than n satisfying $p(A) = 0$.

(8) Let R be a commutative ring (with unit) having only one maximal ideal $\mathfrak{m}$. Show that any element not in $\mathfrak{m}$ is invertible.

(9) Let R be a PID, and M be a finitely generated projective module over R. Show that M is free.

(10) Let R be a Noetherian ring, and let M and N be finitely generated R-modules. Show that the R-module $\mathrm{Hom}_R(M, N)$ is finitely generated.