Algebra Comprehensive exam
May 2016
Department of Mathematics

Instructions:

1. There are 8 questions.
2. There are 100 marks.
3. No books or notes may be used.
4. Unless otherwise indicated, give complete justifications for your solutions. Answers without appropriate reasoning will not receive credit. There will be little or no partial credit – aim for complete solutions.

Question	Points
1	7
2	14
3	8
4	14
5	16
6	12
7	12
8	17
Total:	100

1. (7 points) Consider a finite field $\mathbb{F}_p$ where p is a prime number. Give necessary and sufficient conditions on n so that the field extension $\mathbb{F}_{p^n}/\mathbb{F}_p$ has no proper subextensions.

2. Let p be an odd prime. Let n be an integer not divisible by p. The *Legendre symbol* is defined by

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \equiv x^2 \mod p \text{ for some } x \\ -1 & \text{otherwise} \end{cases}$$

   (a) (4 points) Suppose that n and m are not divisible by p. Show that

$$\left(\frac{nm}{p}\right) = \left(\frac{n}{p}\right)\left(\frac{m}{p}\right).$$

   (b) Write down the formula for $\Phi_p(X)$ the pth cyclotomic polynomial. There is no need to prove your formula is correct.

   (c) (4 points) Let $\zeta \in \mathbb{C}$ be a primitive pth root of unity. Show that

$$1 + \zeta + \zeta^2 + \ldots + \zeta^{p-1} = 0.$$

   (d) (4 points) Show that

$$\sum_{m=1}^{p-1} \left(\frac{m}{p}\right) = 0$$

   where p is an odd prime.

   (e) (2 points) Let ζ be a primitive pth root of unity. Set

$$S = \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) \zeta^n.$$

   Show that

$$S^2 = \left(\frac{-1}{p}\right) p.$$

3. (8 points) Let n be a positive integer and denote by S_{2n} the symmetric group on $2n$-letters. Let D be a Sylow p-subgroup of S_{2n}. Prove or disprove : there is an integer $n > 5$ and a prime p so that D is isomorphic to a dihedral group.

4. Consider the homomorphism of $\mathbb{Z}$-modules

$$\phi : \mathbb{Z}^3 \to \mathbb{Z}^2, \phi(\mathbf{x}) = A\mathbf{x}$$

   where

$$A = \begin{bmatrix} 3 & 9 & 9 \\ 9 & -3 & 9 \end{bmatrix}$$

   (a) (8 points) Find a $\mathbb{Z}$ basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{Z}^2$ and positive integers $d_1 | d_2$ such that $\beta' = \{d_1\mathbf{v}_1, d_2\mathbf{v}_2\}$ is a $\mathbb{Z}$ basis of $\operatorname{im}(\phi)$.

   (b) (6 points) Use the previous part to describe the $\operatorname{coker}(\phi) = \mathbb{Z}^2/\operatorname{im}(\phi)$ as a direct sum of cyclic groups.

5. Let V be a vector space of dimension n over $\mathbb{C}$. In what follows we will write $\operatorname{GL}(V)$ for the group of linear automorphisms of V and for $g \in \operatorname{GL}(V)$ we write $|g|$ for its order in this group An element $g \in \operatorname{GL}(V)$ is called *a pseudo-reflection* if g has finite order (i.e $|g| < \infty$) and the 1-eigenspace of g has dimension $n - 1$. In what follows we will write V_g for the 1-eigenspace of a pseudo-reflection g. (Recall that the 1-eigenspace is the subspace of eigenvectors with eigenvalue 1.)

(a) (4 points) Let G be a finite subgroup of $\mathrm{GL}(V)$ and let V^G be the subspace of vectors fixed by G, that is $v \in V^G$ if and only if $gv = v$ for every $g \in G$. Show that there is a subspace $W \subseteq V$ such that for every $g \in G$, $g(W) = W$ and $W \oplus V^G = V$. (Hint : Consider the linear operator $T : V \to V$ given by

$$T(v) = \frac{1}{|G|} \sum_{g \in G} gv.$$

Consider the image of T and its kernel.)

(b) (4 points) Give an explicit example to show that there exist pseudo-reflections $g, h \in \mathrm{GL}(V)$ with $|g| = |h|$ and $V_g = V_h$ but g and h do not commute.

(c) (4 points) Suppose that g and h are pseudo-reflections with $V_g = V_h$ and the subgroup of $\mathrm{GL}(V)$ generated by g and h is finite then g and h commute. Further, if g and h have the same characteristic polynomial then $g = h$.

(d) (4 points) Suppose that g and h are pseudo-reflections such that $V_g \neq V_h$ and $G =< g, h >$ is finite. If for any other pseudo-reflection $k \in < g, h >$ we have $V_k = V_g$ or $V_k = V_h$ then g and h commute. (Here $< g, h >$ is the subgroup of $\mathrm{GL}(V)$ generated by g and h.)

6. Let V be a complex vector space with a positive definite Hermitian form $< v, w >$. Let T be a self adjoint operator on V.

   (a) (4 points) Show that every eigenvalue of T is real.

   (b) (4 points) Let v and v' be eigenvectors of T with distinct eigenvalues λ and λ' respectively. Show that v and v' are orthogonal.

   (c) (4 points) Given an example of a self adjoint operator T and two distinct non-orthogonal eigenvectors v and v'.

7. Let $f(x) = x^4 - 4x^2 + 2 \in \mathbb{Q}[X]$.

   (a) (6 points) Find a splitting field K of f over $\mathbb{Q}$ and the degree $[K : \mathbb{Q}]$.

   (b) (6 points) Determine the Galois group of f over $\mathbb{Q}$. Determine the action of the generator(s) of the Galois group explicitly on the roots of f.

8. Let F, L, M, K be fields with $F \subset L \subset K$ and $F \subset M \subset K$. Assume that $[L : F] < \infty$ and $[M : F] < \infty$.

   (a) (5 points) Let $\{m_1, \ldots, m_k\}$ be a basis of M as an $L \cap M$ vector space. Show that $E = \sum_{i=1}^{k} Lm_i$, the L subspace of K spanned by $\{m_1, \ldots, m_k\}$ is a subfield of K containing both L and M.

   (b) (4 points) Explain why (a) implies that $[LM : L] \leq [M : L \cap M]$.

   (c) (4 points) Use (a) to show that $[LM : F] = [L : F][M : F]$ implies that $L \cap M = F$. (Hint: draw a picture of all fields involved, including $L \cap M$!)

   (d) (4 points) Let $F = \mathbb{Q}$ and $K = \mathbb{C}$. Let α be a real cube root of 2 and β a complex cube root of 2 and let $L = \mathbb{Q}(\alpha)$ and $M = \mathbb{Q}(\beta)$. Carefully justify that $[LM : F] < [L : F][M : F]$ in this case.