Department of Mathematics  
Western University  

PhD Comprehensive Examination Part I: Analysis  

May 9, 2016  1:00-4:00 p.m.

Instructions: There are eight questions. Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

1. (a) Are the functions

\[ f(z) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \frac{(z - i)^n}{(2 - i)^{n+1}} \]

analytic continuations of each other? Justify your answer.

(b) The series \( h(z) = \sum_{k=0}^{\infty} z^{2^k} = z + z^2 + z^4 + \ldots \) converges for \( |z| < 1 \). (There is no need to prove this.) Show that it cannot be continued analytically beyond the unit disc.

2. Let \((X, d)\) and \((Y, \rho)\) be complete metric spaces. Prove that if \(A\) is a dense subset of \(X\) and \(f : A \to Y\) is an isometry, then \(f\) extends to an isometry \(F : X \to Y\). Give a clear definition of \(F\), prove that it is well-defined, and show it is an isometry.

3. Evaluate \( \int_{0}^{2\pi} \frac{\sin^2 \theta}{5 + 4\cos \theta} d\theta \).

4. Suppose \(f : \mathbb{R}^2 \to \mathbb{R}\) is continuous. Prove that

\[ g(t) = \int_{0}^{t} f(t, x) \, dx \]

defines \(g\) as a continuous function from \(\mathbb{R}\) to \(\mathbb{R}\).

5. Find all solutions of the equation \(e^z = 1 + 2z\) satisfying \(|z| < 1\).
6. Suppose \( y = y(x) \) is the unique solution to the initial value problem,
\[
y'' = xy' + 3y, \quad y(0) = 0, \quad y'(0) = 1.
\]

(a) Prove that \( y \) and all of its derivatives are increasing functions on \([0, 1]\).

(b) Express \( y^{(5)}(1) \) in terms of \( y'(1) \) and \( y(1) \).

(c) Use Taylor’s theorem to show that, for all \( x \in [0, 1] \),
\[
|y(x) - x - \frac{2}{3}x^3| \leq \frac{1}{3}y'(1) + \frac{3}{10}y(1).
\]
(In this question there is no need to give a closed form for the solution \( y(x) \).)

7. Let \( d_1, d_2, \ldots \) be metrics on \( X \) satisfying \( d_k(x, y) \leq 1 \) for each \( k \) and all \( x, y \in X \). Define \( d \) by
\[
d(x, y) = \sum_{k=1}^{\infty} 2^{-k}d_k(x, y).
\]
Then \( d \) is also a metric on \( X \). (There is no need to prove this.) Show that if the metric space \((X, d_k)\) is compact for each \( k \), then the metric space \((X, d)\) is also compact.

8. Let \( h(t, z) \) be a continuous complex-valued function defined for \( t \in [a, b] \) and \( z \in \mathbb{C} \). Suppose that for each fixed \( t \), \( h(t, z) \) is analytic. Show that
\[
H(z) = \int_a^b h(t, z) \, dt
\]
is an entire function.