Department of Mathematics
Western University

PhD Comprehensive Examination Part I: Analysis

May 9, 2016   1:00-4:00 p.m.

Instructions: *There are eight questions. Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.*

1. (a) Are the functions

$$f(z)=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(2-i)^{n+1}}$$

analytic continuations of each other? Justify your answer.

(b) The series $h(z)=\sum_{k=0}^{\infty} z^{2^k} = z+z^2+z^4+\dots$ converges for $|z|<1$. (There is no need to prove this.) Show that it cannot be continued analytically beyond the unit disc.

2. Let (X,d) and (Y,ρ) be complete metric spaces. Prove that if A is a dense subset of X and $f : A \to Y$ is an isometry, then f extends to an isometry $F : X \to Y$. Give a clear definition of F, prove that it is well-defined, and show it is an isometry.

3. Evaluate $\displaystyle\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$.

4. Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ is continuous. Prove that

$$g(t)=\int_0^t f(t,x)\,dx$$

defines g as a continuous function from $\mathbf{R}$ to $\mathbf{R}$.

5. Find all solutions of the equation $e^z = 1+2z$ satisfying $|z|<1$.

6. Suppose $y = y(x)$ is the unique solution to the initial value problem,

$$y'' = xy' + 3y, \quad y(0) = 0, \quad y'(0) = 1.$$

(a) Prove that y and all of its derivatives are increasing functions on $[0, 1]$.

(b) Express $y^{(5)}(1)$ in terms of $y'(1)$ and $y(1)$.

(c) Use Taylor's theorem to show that, for all $x \in [0, 1]$,

$$\left| y(x) - x - \frac{2}{3}x^3 \right| \leq \frac{1}{3}y'(1) + \frac{3}{10}y(1).$$

(In this question there is no need to give a closed form for the solution $y(x)$.)

7. Let $d_1, d_2, \ldots$ be metrics on X satisfying $d_k(x, y) \leq 1$ for each k and all $x, y \in X$. Define d by

$$d(x, y) = \sum_{k=1}^{\infty} 2^{-k} d_k(x, y).$$

Then d is also a metric on X. (There is no need to prove this.) Show that if the metric space (X, d_k) is compact for each k, then the metric space (X, d) is also compact.

8. Let $h(t, z)$ be a continuous complex-valued function defined for $t \in [a, b]$ and $z \in \mathbf{C}$. Suppose that for each fixed t, $h(t, z)$ is analytic. Show that

$$H(z) = \int_a^b h(t, z)\, dt$$

is an entire function.