Algebra Comprehensive exam
September 2017
Department of Mathematics

Instructions:

1. There are 9 questions.
2. No books, notes or other resources may be used.
3. Unless otherwise indicated, give complete justifications for your solutions. Answers without appropriate reasoning will not receive credit. There will be little or no partial credit – aim for complete solutions.

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1. (a) Find a generator for the group of units $(\mathbb{Z}/17\mathbb{Z})^\times$.
   (b) Prove that $\mathbb{Q}^\times$ is not a cyclic group.
2. Explicitly construct a Sylow 2-subgroup in the symmetric group S_6.
3. Let V be a complex vector space. Say a subspace $W \subseteq V$ has *finite codimension* if and only if the quotient space V/W has finite dimension, that is, $[V : W] = \dim_{\mathbb{C}}(V/W)$ is finite. Let $W_1, W_2 \subseteq V$ be subspaces.
   (a) Prove: if W_1, W_2 have finite codimension, then $W_1 \cap W_2$ has finite codimension.
   (b) Show that $[V : W_1 \cap W_2] = [V : W_1] + [W_1 : W_1 \cap W_2]$.
4. Let V be a 2-dimensional, real vector space, and $T\colon V \to V$ an orthogonal linear transformation. Prove that T is diagonalizable over $\mathbb{C}$.
5. Recall that a ring element r is *nilpotent* if $r^n = 0$ for some positive integer n, and *unipotent* if $r - 1$ is nilpotent. Characterize the nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$.
6. Construct *three* examples each (if possible) of upper triangular 3×3 real matrices A, B, C, D satisfying the following. If an example does not exist, briefly explain why.
   (a) A is diagonal and has characteristic polynomial $\lambda^2(\lambda - 1)$.
   (b) B has minimal polynomial $\lambda^2(\lambda - 1)$ and characteristic polynomial $\lambda(\lambda - 1)^2$.
   (c) C is orthogonal but is not a scalar multiple of the identity matrix.
   (d) D is nilpotent and unipotent.
7. Let F be the splitting field of $x^4 + 2x^2 + 2 \in \mathbb{Q}[x]$. Compute the Galois group of the extension $F/\mathbb{Q}$.
8. Suppose A is a real matrix with characteristic polynomial $(\lambda^2 + 1)(\lambda^2 + 2)$. Describe all real subspaces $V \subseteq \mathbb{R}^4$ satisfying $A(V) \subseteq V$.
9. Construct the finite field $\mathbb{F}_{5^2}$, and find an element of the multiplicative group $\mathbb{F}_{5^2}^\times$ which is not a cube. Explain why your constructions are valid.