Department of Mathematics
Western University

PhD Comprehensive Examination Part I: Analysis

October 5, 2017   2:00-5:00 p.m.

Instructions: *There are seven questions. Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.*

1. Suppose $u, v \in (0,1)$.

(a) Prove that x and y are well defined as functions of (u,v) by

$$\sin(ux) = v, \quad 0 < ux < \pi/2, \qquad \text{and} \qquad \sin(uy) = v, \quad \pi/2 < uy < \pi.$$

(b) Sketch the following three subsets of $\mathbb{R}^2$ on the same (large, clearly labeled) xy-axes: $A = \{(x,y) : u = 1/2,\ v \in (0,1)\}$, $B = \{(x,y) : u \in (0,1),\ v = 1/2\}$ and $C = \{(x,y) : u, v \in (0,1)\}$.

2. Let $p(z) = az^n + z + 1$ with $n \geq 2$ and $a \in \mathbb{C}$.

(a) Suppose $|a| < 1/2^n$. Prove that p has exactly one root in the disc $|z| < 2$.

(b) Show that for any $a \in \mathbb{C}$, p has at least one root in the disc $|z| \leq 2$.

3. Let $x_1, x_2, \ldots$ be real numbers and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \inf_{k=1,2,\ldots} k|x - x_k|$.

(a) Show that if the sequence $\{x_n\}_{n=1}^{\infty}$ has no convergent subsequence then f is continuous.

(b) Find a sequence $x_1, x_2, \ldots$ such that f is not continuous.

4. Let k, m, M be positive constants.

(a) Suppose F is a continuous function satisfying $|F(Re^{it})| \leq \frac{M}{R^k}$ when $R > 0$ and $0 \leq t \leq \pi$. Prove that

$$\lim_{R\to\infty} \int_{\Gamma} e^{imz} F(z)\, dz = 0$$

where Γ is the semicircular arc $\{Re^{it} : 0 \leq t \leq \pi\}$.

(b) Show that $\displaystyle\int_0^\infty \frac{\cos(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m}$.

5. Let (M,d) be a metric space and let X be the collection of all Cauchy sequences in M. For $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ in X let $A(x,y) = (x_1, y_1, x_2, y_2, x_3, y_3, \dots)$. We say $x \sim y$ provided $A(x,y) \in X$.

(a) Prove that $\sim$ is an equivalence relation on X.

(b) Fix $x \in X$ and $m \in M$ and let y be the constant sequence $(m, m, m, \dots)$. Show that x converges to m if and only if $x \sim y$.

6. Let Ω be a connected open subset of $\mathbb{C}$ and $f : \Omega \to \Omega$ be a holomorphic map such that $f \circ f = f$. Show that either f is the identity map on Ω or f is constant.

7. For each positive integer n, define $f_n : (0,\infty) \to \mathbb{R}$ by $f_n(x) = \displaystyle\int_0^1 t^{x-1}(1-t)^{n-1}\,dt$.

(a) Prove that for each $x > 0$, $\lim_{n\to\infty} f_n(x) = 0$.

(b) Prove that $\{f_n\}_{n=1}^\infty$ does not converge uniformly to the zero function on $(0,\infty)$.