Instructions: There are seven questions. Answer as many questions as you can. More credit will be given for a complete solution than for several partial solutions.

1. Suppose \( u, v \in (0, 1) \).

(a) Prove that \( x \) and \( y \) are well defined as functions of \((u, v)\) by

\[
\sin(ux) = v, \quad 0 < ux < \pi/2, \quad \text{and} \quad \sin(uy) = v, \quad \pi/2 < uy < \pi.
\]

(b) Sketch the following three subsets of \( \mathbb{R}^2 \) on the same (large, clearly labeled) \( xy \)-axes:

\[
A = \{(x, y) : u = 1/2, \ v \in (0, 1)\}, \quad B = \{(x, y) : u \in (0, 1), \ v = 1/2\} \quad \text{and} \quad C = \{(x, y) : u, v \in (0, 1)\}.
\]

2. Let \( p(z) = az^n + z + 1 \) with \( n \geq 2 \) and \( a \in \mathbb{C} \).

(a) Suppose \( |a| < 1/2^n \). Prove that \( p \) has exactly one root in the disc \( |z| < 2 \).

(b) Show that for any \( a \in \mathbb{C} \), \( p \) has at least one root in the disc \( |z| \leq 2 \).

3. Let \( x_1, x_2, \ldots \) be real numbers and define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = \inf_{k=1,2,\ldots} |k|x - x_k| \).

(a) Show that if the sequence \( \{x_n\}_{n=1}^{\infty} \) has no convergent subsequence then \( f \) is continuous.

(b) Find a sequence \( x_1, x_2, \ldots \) such that \( f \) is not continuous.

4. Let \( k, m, M \) be positive constants.

(a) Suppose \( F \) is a continuous function satisfying \( |F(Re^{it})| \leq \frac{M}{R \pi} \) when \( R > 0 \) and \( 0 \leq t \leq \pi \). Prove that

\[
\lim_{R \to \infty} \int_{\Gamma} e^{imz} F(z) \, dz = 0
\]

where \( \Gamma \) is the semicircular arc \( \{Re^{it} : 0 \leq t \leq \pi\} \).
(b) Show that \[ \int_0^\infty \frac{\cos(mx)}{x^2 + 1} \, dx = \frac{\pi}{2} e^{-m}. \]

5. Let \((M, d)\) be a metric space and let \(X\) be the collection of all Cauchy sequences in \(M\). For \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\) in \(X\) let \(A(x, y) = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)\). We say \(x \sim y\) provided \(A(x, y) \in X\).

(a) Prove that \(\sim\) is an equivalence relation on \(X\).

(b) Fix \(x \in X\) and \(m \in M\) and let \(y\) be the constant sequence \((m, m, m, \ldots)\). Show that \(x\) converges to \(m\) if and only if \(x \sim y\).

6. Let \(\Omega\) be a connected open subset of \(\mathbb{C}\) and \(f : \Omega \to \Omega\) be a holomorphic map such that \(f \circ f = f\). Show that either \(f\) is the identity map on \(\Omega\) or \(f\) is constant.

7. For each positive integer \(n\), define \(f_n : (0, \infty) \to \mathbb{R}\) by \(f_n(x) = \int_0^1 t^{x-1}(1 - t)^{n-1} \, dt\).

(a) Prove that for each \(x > 0\), \(\lim_{n \to \infty} f_n(x) = 0\).

(b) Prove that \(\{f_n\}_{n=1}^\infty\) does not converge uniformly to the zero function on \((0, \infty)\).