Algebra Comprehensive exam
May 2017
Department of Mathematics

Instructions:

1. There are 8 questions.
2. There are 100 marks.
3. No books or notes may be used.
4. Unless otherwise indicated, give complete justifications for your solutions. Answers without appropriate reasoning will not receive credit. There will be little or no partial credit – aim for complete solutions.

| Question | Points |
|---|---|
| 1 | 12 |
| 2 | 12 |
| 3 | 14 |
| 4 | 12 |
| 5 | 12 |
| 6 | 12 |
| 7 | 14 |
| 8 | 12 |
| Total: | 100 |

1. (12 points) Show that $\mathbb{Z}[X]$ is not a principal ideal domain.

> **Solution:** The ideal $< 2, X >$ is proper, (why? students must give reason!). Suppose that $< 2, X >=< f >$. Then $rf = X$ and $X$ is irreducible. By properness we must have $f = X$. But $X$ does not divide 2.

2. (12 points) Let $F$ be a field and let $M$ be an invertible $2 \times 2$ matrix with entries in $F$. Suppose that there is a positive integer so that $M^n = I_2$. Prove or disprove : $M$ is diagonalisable.

> **Solution:** The statement is false in positive characteristic.

3. (14 points) Consider the group homomorphsim

$$\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$$

given by left multiplication by the matrix

$$\begin{bmatrix} 102 & 69 \\ 48 & 33 \end{bmatrix}.$$

What is $|\mathrm{coker}(\phi)|$? (Here $|X|$ means order (cardinality).)

> **Solution:** After suitable choice of basis (row/col ops) the matrix of the homomorphism is
>
> $$\begin{bmatrix} 3 & 0 \\ 0 & 18 \end{bmatrix}$$
>
> The cokernel can now be read off.

4. (12 points) Consider the ring $V = \mathbb{C}[x]/(x^4 + 2x^2 + 1)$. We can view $V$ as a finite dimensional vector space and mulitplication by $x$ gives a linear operator $V \to V$. Find the Jordan form of this operator.

> **Solution:** Follows easily from chinese remainder theorem.

5. (12 points) Let $G$ be a group of order $p^n$ where $p$ is a prime and $n > 0$. Suppose that $G$ is simple. Show that $n = 1$.

> **Solution:** Class equation gives non-trivial centre. We reduce to $G$ abelian and apply the classification.

6. (12 points) Let $R$ be an integral domain. Define what it means for $x \in R$ to be irreducible. Now suppose that $x$ is irreducible. Prove or disprove : the ideal $< x >= xR$ is prime.

> **Solution:** Irreducible means that $x$ is not zero nor a unit and if $x = ab$ then one of $a$ or $b$ is a unit. The assertion is false. Take $R = k[t^2, t^3]$. In this ring $t^3$ is irreducible. But the ideal it generates is not prime, indeed $(t^2)^2 \in < t^3 >$ but $t^2 \notin < t^3 >$.

7. (14 points) Let $F = \mathbb{Q}(\sqrt{2}) := \{q_1 + q_2\sqrt{2} \mid q_1, q_2 \in \mathbb{Q}\}$. Determine which field extensions $F_1/\mathbb{Q}, F_2/\mathbb{Q}_2, F_3/\mathbb{Q}$ described below, are Galois, and in the case when they are Galois, determine their Galois groups. Recall that $F(\alpha)$ means the smallest field containing $F$ and $\alpha$.

   (1) $F_1 = F(\sqrt[4]{2})$.

   (2) $F_2 = F(2 + \sqrt{2})$.

   (3) $F_3 = F(1 + \sqrt{2})$.

   **Solution:** $F_2$ is the only Galois extension and $\mathrm{Gal}(F_2/\mathbb{Q})$ is a cyclic group of order 4. In cases (1) and (3) the number of automorphisms $F_i/\mathbb{Q}$ is smaller than $[F_i : \mathbb{Q}]$.

8. (12 points) Show that any group of order 21 contains a normal cyclic subgroup of order 7.

   Bonus Question: Determine all groups of order 21.

   **Solution:** The Sylow theorem tells us that the number $C_7$ is $1 + k.7$ and $1 + k.7$ divides 21. Hence it is only one. Further analysis of the possible action of $C_3$ on $C_7$ shows that up to isomorphism, there are only two groups of order 21.