Ph.D. Comprehensive Exam, Analysis

This exam has 8 problems. Each problem is worth 10 points.
Carefully explain each step of your solution. Complete solution
to one problem may well be worth more than partial solutions
to several problems.

Real Analysis

1. Let U be an open neighbourhood of $0 \in \mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be Lipschitz continuous, with a Lipschitz constant $K > 0$. Let $0 < a < 1$ be such that the closed ball $\overline{B}_{2a}(0)$ is contained in U and the norm $\|f(x)\| \leq L$ for some constant $L > 0$ and all $x \in \overline{B}_{2a}(0)$. Let $b > 0$ be such that $b < \min\{\frac{a}{L}, \frac{1}{K}\}$.
   (a) For a point $x \in \overline{B}_a(0)$, let M_x be the set of continuous maps $\alpha : [-b, b] \to \overline{B}_{2a}(0)$ satisfying $\alpha(0) = x$. For $\alpha \in M_x$, define $S_x(\alpha)$ to be the map
   $$[-b, b] \ni t \mapsto x + \int_0^t f(\alpha(u))\mathrm{d}u \in \mathbb{R}^n\,.$$
   Show that S_x maps M_x into M_x, and it is a contraction.
   (b) Show that, for every $x_0 \in \overline{B}_a(0)$, there exists a unique $\alpha_0 \in M_{x_0}$ satisfying
   $$\alpha_0(t) = x_0 + \int_0^t f(\alpha_0(u))\,\mathrm{d}u$$
   (i.e., a unique local solution to the initial value problem $x'(t) = f(x(t))$, $x(0) = x_0$).

3. Let $\Phi : (\mathbb{R}^n)^n \to \mathbb{R}$ be given as $\Phi(v_1, \ldots, v_n) = \det\left[v_j^i\right]_{i,j=1,\ldots,n}$, where $v_j = (v_j^1, \ldots, v_j^n) \in \mathbb{R}^n$ for $j = 1, \ldots, n$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis in $\mathbb{R}^n$, and let $h_j = (j, \ldots, j) \in \mathbb{R}^n$ for $j = 1, \ldots, n$. Evaluate $D\Phi(e_1, \ldots, e_n).(h_1, \ldots, h_n)$. Justify your answer.

2. Let $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x^2 + y^2 + z^2 \leq 4\}$. Evaluate
   $$\int_D \sin(xy) - \sin(xz) + \sin(yz)\;\mathrm{d}x\;\mathrm{d}y\;\mathrm{d}z\,.$$
   Justify your answer.

4. (a) Show that the area of a planar region delimited by a closed simple smooth curve C is given by
   $$\frac{1}{2}\int_C x\mathrm{d}y - y\mathrm{d}x\,.$$
   (b) Compute $\int_C (xy - y^2)\mathrm{d}x + (x^2 + 3xy)\mathrm{d}y$, where C is the boundary of the bounded region delimited by the graphs of $y = x^3$ and $x = y^2$.

Complex Analysis

5. Evaluate the following integral
$$\int_{\partial\Omega} \frac{e^{\pi z}}{2z^2 - i}\, dz,$$
where the domain Ω is given by
$$\Omega = \{z \in \mathbb{C} : |z| < 1,\ \operatorname{Im} z > 0,\ \operatorname{Re} z > 0\}.$$

6. Let $f(z)$ be an entire function that satisfies the inequality
$$|f(z)| \leq A + B|z|^k, \quad z \in \mathbb{C},$$
where $A, B > 0$ and k is a positive integer. Prove that $f(z)$ is a polynomial of degree at most k.

7. Suppose that $g(z)$ is a function that is holomorphic in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ and continuous on its closure, $\overline{\mathbb{D}}$. Assume that $\operatorname{Im} g(z) \equiv 0$ on the unit circle $\partial\mathbb{D}$. Prove that $g(z)$ is a constant function.

8. Suppose that a sequence $f_n(z)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ converges to a function $f(z)$ uniformly on compacts in Ω. Prove that $f(z)$ is also a holomorphic function on Ω.