THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

September 2018 | 3 hours

Instructions: There will be little or no partial credit, so you should aim to solve some problems completely and correctly rather than attempting every problem. It is recommended that you attempt at least one question from each topic. Skim the questions at the start, so you can focus on the ones you feel most confident about. Justify all your answers.

Linear Algebra

(1) Let $A \in \mathbb{C}^{6\times 6}$. Assume that
$$\dim\ker(A-2)^2 = 3 \qquad \text{and} \qquad \dim\ker(A-3)^3 = 2.$$
What are the possible Jordan normal forms of A?

(2) Let V be a vector space with only finitely many elements. Compute the sum of all vectors in V.

Rings and modules

(3) Give an example of an integral domain that is not a UFD. (Justify!)

(4) Let $n \in \mathbb{N}$ and let p be a prime. Let V be the vector space of univariate polynomials of degree at most n with coefficients in $\mathbb{F}_p$. We consider V as an $\mathbb{F}_p[X]$-module by letting X act as (formal) differentiation, $X \cdot f = f'$. Determine the primary decomposition of V.

Group theory

(5) Let $Z(G)$ denote the centre of a group G, and let p be a prime.
   (a) Show that if $G/Z(G)$ is cyclic, then G is abelian.
   (b) Use the Class Equation to deduce that every group G of order p^2 is abelian.

(6) Let A_n denote the alternating subgroup of the symmetric group S_n.
   (a) What is the maximal order of an element in S_7?
   (b) What is the maximal order of an element in A_7?

Field theory

(7) Let F be a finite field of characteristic p.
   (a) Prove that F is perfect; that is, every element in F is a p^{th} power in F.
   (b) Prove every irreducible polynomial f over F is separable. Hint: consider f'.

(8) Find the Galois group G (up to isomorphism) of $x^6 - 4x^3 + 4 \in \mathbb{Q}[x]$.