THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 3, 2018 *3 hours*

Instructions: There will be little or no partial credit, so you should aim to solve some problems completely and correctly rather than attempting every problem. The exam is divided into the Real and Complex analytic parts (problems 1–4 and 5–8, respectively). You should attempt *at least* two problems from each group.

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1. Suppose $f(x)$ is a function continuous on $[0,1]$, and differentiable on $(0,1)$. Suppose that $f(0) = 0$, and $\int_0^1 f(x)dx = 1$. Prove that there exists a point $x_0 \in (0,1)$ such that $f'(x_0) > 1$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant function. A number $c \in \mathbb{R}$ is called a *period* of f, if $f(x+c) = f(x)$ for all $x \in \mathbb{R}$. The function f is then called periodic if there exists $p > 0$ such that $f(x+p) = f(x)$ for all $x \in \mathbb{R}$.
   (a) Show that the set of all periods of a given function f forms a subgroup of $(\mathbb{R}, +)$.
   (b) Give an example of a non-constant function f for which the group of periods is not discrete.
   (c) Prove that if $f \neq$ const is continuous, then the group of periods is discrete.

3. Suppose that f is a real-valued function defined on an open subset $\Omega \subset \mathbb{R}^n$ and that the partial derivatives $\partial f/\partial x_j$ exist and are bounded for $j = 1, \ldots, n$. Prove that f is continuous on Ω.

4. Let X denote the space of all sequences of real numbers. Let $x = (x_k)_{k=1}^{\infty}$ and $y = (y_k)_{k=1}^{\infty}$ be arbitrary elements of X. Define

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{k^2} \min\{|x_k - y_k|, 1\}.$$

   Prove that (X, d) is a metric space.

5. Prove or give a counterexample to the following statement:

   There is no non-zero polynomial $P(z)$ such that $P(z) \cdot e^{1/z}$ is an entire function.

6. Let γ be the curve $r = \frac{3}{2} + 3\cos\theta$, $\theta \in \mathbb{R}$, traversed one time counterclockwise. Evaluate the integral

$$\int_\gamma \frac{\sin(\pi z)}{z^2 - 3z + 2}\, dz\,.$$

7. Let $f(z) = \dfrac{1}{z} - \dfrac{1}{z^2+1}$. Find all possible Laurent expansions of f about $z_0 = i$ and determine where each is valid.

8. Let $(z_n)_{n=1}^{\infty}$ be a sequence of distinct complex numbers such that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|z_n|^3}$ converges, and let

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{1}{(z-z_n)^2} - \frac{1}{z_n^2} \right)\,.$$

   Prove that f is meromorphic on $\mathbb{C}$ and find all its poles.