UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

May 2018 | 3 hours

Instructions:

- There are 8 questions in total. Answer *completely* as many questions as you can.
- **All answers must be justified** unless stated otherwise.
- They will be graded according to correctness, completeness, and clarity of presentation. More credit may be given for a complete solution than for several partial solutions.

Good luck!

(1) Let R be a commutative ring with 1 and I an ideal in R. Show that there exists a minimal prime ideal P such that $I \subseteq P$.

(2) Let R be a commutative ring with 1. Let P and Q be prime ideals of R and suppose that every element of $R \setminus (P \cup Q)$ is a unit. Show that either P or Q is maximal.

(3) Let λ be an eigenvalue of an $n \times n$ complex matrix A with algebraic multiplicity k. Show that the matrix $(A - \lambda I)^k$ is of rank $n - k$.

(4) For $\lambda \in \mathbb{R}$, we define a symmetric bilinear form $\langle -, - \rangle$ on the space of all 2×2 real matrices by

$$\langle A, B \rangle = \lambda \cdot \operatorname{tr}(A \cdot B) + \operatorname{tr}(A \cdot B^t),$$

where $\operatorname{tr} A$ denotes the trace of a matrix A and A^t is the transpose of A. For which values of $\lambda \in \mathbb{R}$ is the form $\langle -, - \rangle$ positive-definite?

(5) Let G be a finite group and H a subgroup of index p where p is the smallest prime dividing the order of G. Prove that H is a normal subgroup of G.

(6) Let p be a prime number and let S_p be the symmetric group on p letters. Let τ_p be a p-cycle in S_p. Determine the size of the normaliser subgroup of $C_p = \langle \tau_p \rangle$ in S_p.

(7) Determine the Galois group of the polynomial $f(x) = x^8 - 1$ over the finite field $\mathbb{F}_3$.

(8) Let F, K, L be fields where K/F and L/F are Galois extensions. Show that the composite KL is Galois over F and that $\operatorname{Gal}(KL/F)$ is isomorphic to the following subgroup of $\operatorname{Gal}(K/F) \times \operatorname{Gal}(L/F)$:

$$\{(\sigma, \tau) \in \operatorname{Gal}(K/F) \times \operatorname{Gal}(L/F) : \sigma|_{K \cap L} = \tau|_{K \cap L}\}$$