THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 4, 2018 *3 hours*

Instructions: Answer as many questions as you can. Give complete justifications. Aim for complete solutions (there will be little or no partial credit).

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1. Suppose f is a holomorphic function on a neighborhood of the closed unit disk, such that $|f(z)| \geq 2$ on the unit circle and $f(0) = 1$. Show that f has a zero in the unit disk.

2. Let C be the circle in the complex plane that has radius 3 and centre 0, traced once in the counterclockwise direction. Calculate

$$\int_C \frac{e^z}{z^4 + z^2}\, dz.$$

3. Let f be a function continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} - \{z \in \mathbb{C} \mid \operatorname{Re} z = 0\}$. Prove that f is entire.

4. How many zeroes does the function $f(z) = \dfrac{1}{10}e^z - z$ have in the annulus $1 < |z| < 2$?

5. Show that the following limit exists:

$$\lim_{x\to\infty} \int_0^x \cos(t^3 + t)dt.$$

6. How many terms of the series

$$\sum_1^\infty \frac{1}{n^2}$$

you have to add in order to approximate its value to within $\frac{1}{10}$? (No need to give the best possible answer).

7. Let $0 \leq \alpha < 1$ be a constant. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x - \alpha \sin x, \qquad x \in \mathbb{R}.$$

Show that f is one to one and onto and its inverse function is smooth.

8. Let $\Delta = \{(p_1, \cdots p_n) \in \mathbb{R}^n;\ p_i \geq 0,\ \sum p_i = 1\}$. Consider the function $S : \Delta \to \mathbb{R}$ defined by

$$S(p_1, \cdots, p_n) = -\sum_{i=1}^n p_i \ln p_i.$$

(We define $0 \ln 0 = 0$). What is the maximum value of S and where is it obtained? Prove your claim.