Linear Algebra.

(1) Let $V$ and $W$ be vector spaces over the field $F$ with $\dim V = m$ and $\dim W = n$. Let $f: V \to W$ be a linear map. Show that there are ordered bases $\beta$ for $V$ and $\gamma$ for $W$ and a unique $r \geq 0$ such that the matrix representing $f$ with respect to $\beta$ and $\gamma$ is the block matrix

$$\begin{bmatrix}
I_r & O_{m-r,m-r} \\
O_{n-r,n-r} & O_{n-r,m-r}
\end{bmatrix}$$

where $I_r$ is the $r \times r$ identity matrix and $O_{p,q}$ the zero matrix of size $p \times q$.

(2) Let $A$ be an $n \times n$ matrix over $\mathbb{C}$ of rank 1. Show that $\det(A + I) = \text{tr}(A) + 1$.

Rings and modules.

(3) Show that a principal ideal domain is Noetherian.

(4) Let $R$ be a PID. Let $f: M \to N$ and $g: N \to M$ be morphisms between two free $R$-modules of the same finite rank $n$. Show that if $gf$ is the identity of $M$, then $f$ and $g$ are isomorphisms.

Group theory.

(5) Let $p$ be a prime. Show that a finite $p$-group is nilpotent.

(6) Construct a non-abelian group of order 42.

Field theory.

(7) Let $\zeta \in \mathbb{C}$ satisfy $\zeta^7 = 1 \neq \zeta$. What is the minimal polynomial of $\theta = \zeta + \zeta^{-1}$ over $\mathbb{Q}$?

(8) Find $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$ with group isomorphic to $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Do not forget to justify your answers!