UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

October 2019 3 hours

Instructions: Answer completely as many questions as you can. More credit may be given for a complete solution than for several partial solutions. Do not forget to justify your answers.

Linear Algebra.

(1) Let V and W be vector spaces over the field F with $\dim V = m$ and $\dim W = n$. Let $f\colon V \to W$ be a linear map. Show that there are ordered bases β for V and γ for W and a unique $r \geq 0$ such that the matrix representing f with respect to β and γ is the block matrix

$$\begin{bmatrix} I_r & O_{m-r,m-r} \\ O_{n-r,n-r} & O_{n-r,m-r} \end{bmatrix}$$

where I_r is the $r \times r$ identity matrix and $O_{p,q}$ the zero matrix of size $p \times q$.

(2) Let A be an $n \times n$ matrix over $\mathbb{C}$ of rank 1. Show that $\det(A + I) = \operatorname{tr}(A) + 1$.

Rings and modules.

(3) Show that a principal ideal domain is Noetherian.

(4) Let R be a PID. Let $f\colon M \to N$ and $g\colon N \to M$ be morphisms between two free R-modules of the same finite rank n. Show that if gf is the identity of M, then f and g are isomorphisms.

Group theory.

(5) Let p be a prime. Show that a finite p-group is nilpotent.

(6) Construct a non-abelian group of order 42.

Field theory.

(7) Let $\zeta \in \mathbb{C}$ satisfy $\zeta^7 = 1 \neq \zeta$. What is the minimal polynomial of $\theta = \zeta + \zeta^{-1}$ over $\mathbb{Q}$?

(8) Find $\alpha \in \mathbb{C}$ such that $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$ with group isomorphic to $G = \mathbb{Z}/2 \times \mathbb{Z}/2$.

Do not forget to justify your answers!