THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 8, 2019 *3 hours*

Instructions: Answer as many questions as you can. Give complete justifications. Aim for complete solutions (there will be little or no partial credit).

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1. Let S be a set of strictly positive real numbers and assume there is a constant C such that

$$\sum_{x\in J} x \leq C$$

for all finite subsets J of S. Show that S is countable.

2. Prove the Banach fixed point theorem: If (X,d) is a complete metric space and $f : X \to X$ is a contractive map in the sense that there is a constant $k < 1$ such that

$$d(f(x), f(y)) \leq k\, d(x,y) \qquad \forall x, y \in X,$$

then f has a unique fixed point. Show that the theorem can fail if X is not complete.

3. Show that for any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with compact support we have

$$\int_{\mathbb{R}^2} (\Delta f(x)) \ln |x| d^2x = 2\pi f(0),$$

where $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator and $|x|$ is the Euclidean norm of x. (Hint: Try to use Green's theorem).

4. Using basic calculus, show that the following limit exits:

$$\lim_{R\to\infty} \int_0^R \sin(x^3) dx.$$

5. Suppose f is an entire function with the following property: for each $z \in \mathbb{C}$ there is $n \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. Prove that f is a polynomial.

6. Use contour integral techniques to evaluate

$$\int_0^\infty \frac{x^2}{(x^2+1)^2(x^2+9)}\, dx.$$

7. Use Rouché's Theorem to prove that the equation

$$e^z = z + 2$$

has exactly one solution in the half plane $\{z \in \mathbb{C} \mid Re(z) < 0\}$. Also, show that this solution is real.

8. Suppose f is an entire function such that $f(0) = 4 - 2i$ and $|f(z)| \leq 2\sqrt{5}$ for $|z| \leq 1$. Find $f'(0)$.