1. Let $S$ be a set of strictly positive real numbers and assume there is a constant $C$ such that
\[ \sum_{x \in J} x \leq C \]
for all finite subsets $J$ of $S$. Show that $S$ is countable.

2. Prove the Banach fixed point theorem: If $(X, d)$ is a complete metric space and $f : X \to X$ is a contractive map in the sense that there is a constant $k < 1$ such that
\[ d(f(x), f(y)) \leq k d(x, y) \quad \forall x, y \in X, \]
then $f$ has a unique fixed point. Show that the theorem can fail if $X$ is not complete.

3. Show that for any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ with compact support we have
\[ \int_{\mathbb{R}^2} (\Delta f(x)) \ln |x| d^2x = 2\pi f(0), \]
where $\Delta = \partial_1^2 + \partial_2^2$ is the Laplace operator and $|x|$ is the Euclidean norm of $x$. (Hint: Try to use Green’s theorem).

4. Using basic calculus, show that the following limit exits:
\[ \lim_{R \to \infty} \int_0^R \sin(x^3)dx. \]

5. Suppose $f$ is an entire function with the following property: for each $z \in \mathbb{C}$ there is $n \in \mathbb{N}$ such that $f^{(n)}(z) = 0$. Prove that $f$ is a polynomial.

6. Use contour integral techniques to evaluate
\[ \int_0^\infty \frac{x^2}{(x^2 + 1)^2(x^2 + 9)} \, dx. \]

7. Use Rouché’s Theorem to prove that the equation
\[ e^z = z + 2 \]
has exactly one solution in the half plane \( \{ z \in \mathbb{C} \mid \text{Re}(z) < 0 \} \). Also, show that this solution is real.

8. Suppose \( f \) is an entire function such that \( f(0) = 4 - 2i \) and \( |f(z)| \leq 2\sqrt{5} \) for \( |z| \leq 1 \). Find \( f'(0) \).