Algebra Comprehensive exam
May 2019
Department of Mathematics

Instructions:

1. There are 8 questions.
2. There are 80 marks.
3. No books or notes may be used.
4. Unless otherwise indicated, give complete justifications for your solutions. Answers without appropriate reasoning will not receive credit. There will be little or no partial credit – aim for complete solutions of about 5 problems.

Question	Points
1	10
2	10
3	10
4	10
5	10
6	10
7	10
8	10
Total:	80

1. (10 points) Let G be a simple group, we denote its order by $|G|$. Suppose that $1 < |G| \leq 30$. Show that $|G| = p$ a prime number.

2. (10 points) Consider the $\mathbb{Z}$-module homomorphism
$$f : \mathbb{Z}^3 \to \mathbb{Z}^3$$
given by the matrix
$$\begin{bmatrix} -7 & -8 & -8 \\ 4 & 4 & 4 \\ 12 & 0 & 12 \end{bmatrix}$$
Find $|\operatorname{coker}(f)|$?

3. (10 points) Let F be a finite field with 5 elements. Let S be the collection of similarity classes of matrices over F with minimal polynomial $(x-1)(x^2+x+1)^2$ and characteristic polynomial $(x-1)^2(x^2+x+1)^4$. What is $|S|$?

4. (10 points) Assume that K/F is a finite Galois extension, $G = \operatorname{Gal}(K/F)$ is its Galois group, and $\alpha \in K^\times := K \setminus \{0\}$. Also assume char $F \neq 2$. (This means that $1 + 1 \neq 0$ in F.)

   Show that $K(\sqrt{\alpha})/F$ is also a Galois extension iff $\frac{\sigma(\alpha)}{\alpha} \in K^{\times 2}$ for all $\sigma \in G$.

5. (10 points) Consider $K = \mathbb{Q}(\sqrt{5})$ and $L = \mathbb{Q}(\sqrt{7})$. Decide which of these fields can be embedded into a Galois extension $N/\mathbb{Q}$ such that $\operatorname{Gal}(N/\mathbb{Q})$ is a cyclic group of order 4.

6. (10 points) Let $GL_3(\mathbb{F}_p)$ be a group of invertible matrices 3×3 over $\mathbb{F}_p$. ($\mathbb{F}_p$ is a finite field with p-elements, p is a prime number.) Consider matrices $U_3(\mathbb{F}_p)$ which are upper triangular and have all elements on diagonal 1. (Below the diagonal, all elements are zero. On the diagonal, all elements are 1, and above the diagonal, any entries from $\mathbb{F}_p$ are possible.)

   Show that $U_3(\mathbb{F}_p)$ is a p-Sylow subgroup of $GL_3(\mathbb{F}_p)$.

7. (10 points) Show that in $\mathbb{Z}[\sqrt{7}] = \{a + b\sqrt{7} \mid a, b \in \mathbb{Z}\}$.

   a.) Observe that 9 factors as 3.3 but it also is equal to $(4+\sqrt{7})(4-\sqrt{7})$ (this is easy).

   b.) All elements $3, 4+\sqrt{7}, 4-\sqrt{7}$ are irreducible.

   This means that if for example $4+\sqrt{7} = w \cdot W, w, W \in \mathbb{Z}[\sqrt{7}] \Rightarrow w/1$ or $W/1$ (w divides 1 or W divides 1).

   c) Show that $(8+3\sqrt{7})^d \in \mathbb{Z}[\sqrt{7}]^\times$ for each $d \in \mathbb{Z}$. In other words, show that $(8+3\sqrt{7})^d$ is a unit in $\mathbb{Z}[\sqrt{7}]$ for every integer d.

8. (10 points) Let D be a dihedral group of order 8. Draw a lattice of all subgroups of D.