Department of Mathematics, Western University

PhD Comprehensive Examination Part I: Analysis

May 7, 2019 1:00–4:00 p.m.

Instructions: *There are eight questions. Focus on providing complete solutions to the questions you attempt. More credit will be given for a complete solution than for several partial solutions.*

1. Let z be a complex variable and set $f(z) = \sum_{n\geq 0} c_n z^n$ where the coefficients are the Fibonacci numbers defined recursively by $c_0 = c_1 = 1$ and $c_{n+2} = c_{n+1} + c_n$.

(a) Show that $f(z) = \dfrac{1}{1-z-z^2}$ on any disc $D(0,R)$ on which the series converges.

(b) Find the radius of convergence of the series.

2. Suppose $f : [0,1] \to \mathbb{R}$ is continuous, and $a, b \in \mathbb{R}$. Prove that the boundary value equation $u'' = f$, $u(0) = a$, $u(1) = b$ has a unique solution $u \in C^2([0,1])$ given by,

$$u(x) = (1-x)\Big(a - \int_0^x t f(t)\, dt\Big) + x\Big(b - \int_x^1 (1-t) f(t)\, dt\Big).$$

3. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\, d\theta$.

4. Let d_1 and d_2 be metrics on a set X. Consider the following statements.

P: For any metric space (Y,ρ) and any continuous $f\ (X,d_1) \to (Y,\rho)$, the function $f\ (X,d_2) \to (Y,\rho)$ is also continuous.

Q: For any metric space (Y,ρ) and any continuous $f\ (Y,\rho) \to (X,d_2)$, the function $f\ (Y,\rho) \to (X,d_1)$ is also continuous.

Prove that P is true if and only if Q is true.

5. Suppose f is holomorphic in a disc centered at the origin, that $f(0) = 0$, and that $f'(0) \neq 0$. From the inverse function theorem, we know f has an inverse g defined in some neighbourhood of 0. Show that on some open disc $D(0,\varepsilon)$, g is given by

$$g(z) = \frac{1}{2\pi i} \int_{|w|=\varepsilon} \frac{w f'(w)}{f(w) - z}\, dw.$$

6. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function such that $\varphi(x) = 0$ when $|x| \geq 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. For each $x \in \mathbb{R}$, define $g(x) = \int_{-\infty}^{\infty} f(x-t)\varphi(t)\, dt$. Prove that g is differentiable on $\mathbb{R}$.

7. Let f be a holomorphic function defined on the open unit disc. Suppose there exists an open arc R on the unit circle having the property that $\lim f(z) = 1$, as z approaches R (z is in the open unit disc). Prove that f is identically 1.

8. Let d be a positive integer and consider the sequence $(f_n)_{n=1}^{\infty}$, where $f_n : \mathbb{R}^d \to [0,1]$ for $n = 1, 2, \ldots$. Prove that there is a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that for each $q \in \mathbb{Q}^d$, $\lim_{k\to\infty} f_{n_k}(q)$ converges.