PhD Comprehensive Exam (Algebra)
Department of Mathematics
September 2020

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. You should attempt at least one question from each topic.
4. Justify all your answers.

Linear Algebra

1. Let V be the vector space of all 2×2 matrices with real entries, and let W be the subspace spanned by all matrices of the form $[A, B] = AB - BA$, where $A, B \in V$. What is the dimension of W?

2. Let n be a positive integer, let V be an n-dimensional vector space over a field K, and let $S, T : V \to V$ be linear transformations. Show that $S \circ T - T \circ S = \mathrm{id}_V$, where id_V denotes the identity transformation $V \to V$, is impossible when the characteristic of K is zero.

Groups

3. Let D_{103} denote the dihedral group of the regular 103-gon. Find the number of ordered commuting pairs of elements in D_{103}; that is, find the cardinality of the set

$$X = \{(a, b) \in D_{103} \times D_{103} \, : \, ab = ba\}.$$

4. Let p be a prime number and let $G = \mathrm{GL}_2(\mathbb{F}_p)$, where $\mathbb{F}_p$ is the field with p-many elements. How many Sylow p-subgroups does G have?

Rings and Modules

5. Let R be a UFD, and let r be any nonzero, nonunit irreducible element of R. Prove that $R/(r)$ is also a UFD or disprove by counterexample. What happens when R is a PID: is $R/(r)$ also a PID?

6. Let R be a commutative ring with identity. Let M be a Noetherian R-module. Let $\phi : M \to M$ be an epimorphism of R-modules. Prove or disprove: ϕ is an isomorphism.

Fields

7. In this question $p \in \mathbb{Z}$ is a prime number. Let $K = \mathbb{F}_p(t)$ be the field of fractions of the polynomial ring $\mathbb{F}_p[t]$. Let $L = K[X]/\langle X^p - t \rangle$. Consider the polynomial ring $L[Y]$. Prove or disprove: the polynomial $Y^p - t^2 + 1$ is irreducible in this ring.

8. Consider $\sqrt{2} \in \mathbb{Q}(\sqrt[4]{2})$. Show that $\sqrt{2}$ is not an 11th power in this field, in other words there is no $\alpha \in \mathbb{Q}(\sqrt[4]{2})$ with $\alpha^{11} = \sqrt{2}$. (Hint: use the field norm.)