THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

September 2, 2020, 1-4 PM

Instructions: Answer as many questions as you can. Give complete justifications. Aim for complete solutions (there will be little or no partial credit).

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1. Let $f : [a, b] \to \mathbb{R}$ be a *convex* function. This means for all $a \leq x_1 < x_2 \leq b$, and all $t \in [0, 1]$ we have
$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2).$$
Show that f is continuous at all points of its domain and also has a right and left derivative at all points.

2. Give examples of metrics d_1 and d_2 on the same set X such that d_1 is a complete metric, d_2 is not complete, but d_1 and d_2 define the same topology on X.

3. Compute the Gaussian integral
$$\int_{\mathbb{R}^2} xye^{-(x^2+2xy+2y^2)}dxdy.$$
(You may use the formula $\int_{\mathbb{R}} e^{-\pi x^2}dx = 1$.)

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic C^2 function. The *Fourier coeficients* of f are defined as
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx, \quad n \in \mathbb{Z}.$$
Show that the series
$$\sum_{n\in\mathbb{Z}} n^2|a_n|^2$$
is convergent.

5. Evaluate
$$\int_{-\infty}^{\infty} \frac{e^{ix}dx}{1+x^2}.$$

6. Prove that if an entire function f satisfies $f(z+1) = f(z)$ and $f(z+i) = f(z)$ for all $z \in \mathbb{C}$, then f is a constant function.

7. Let $\mathbb{D}$ denote the unit disk. Suppose f is function that is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Suppose further that $\mathrm{Re}f$ vanishes on the unit circle. Give the best possible description of the function f.

8. Let $S = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Let $g(z) = \overline{z}$. Prove that $g|_S$ cannot be approximated by holomorphic polynomials uniformly on S.