UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

PH.D. COMPREHENSIVE EXAMINATION (ALGEBRA)

October 2021 3 hours

Instructions: Answer completely as many questions as you can. More credit may be given for a complete solution than for several partial solutions. Do not forget to justify your answers.

Linear Algebra.

(1) Let V be a finite-dimensional complex vector space with a Hermitian inner product. Let $T : V \to V$ be a normal linear transformation. Show T and TT^* have the same rank.

(2) Let A be an $n \times n$ matrix with coefficients in $\mathbb{C}$ of rank one. Show $\det(A + I) = \mathrm{tr}(A) + 1$.

Rings and modules.

(3) Let $R = \mathbb{Z}[T, T^{-1}]$ be the ring of Laurent polynomials in one variable.

(a) Show the units in R are $R^\times = \{ \pm T^n \mid n \in \mathbb{Z} \}$.

(b) Find all ring homomorphisms $f\colon R \to R$ satisfying $f(1) = 1$.

(4) Let M and N be two finitely generated modules over a Noetherian ring R. Let $\mathrm{Hom}_R(M, N)$ be the R-module of R-module homomorphisms $M \to N$. Show $\mathrm{Hom}_R(M, N)$ is finitely generated.

Group theory.

(5) Show the multiplicative group $(\mathbb{Z}/35)^\times$ is not cyclic. Find a pair of elements generating it.

(6) Let G be a group. Recall that a subgroup $H \subseteq G$ is *characteristic* iff $\alpha(H) \subseteq H$ for every automorphism $\alpha\colon G \to G$. Show the center $Z(G)$ and derived subgroup $[G, G]$ are both characteristic.

Field theory.

(7) Determine the number of irreducible quadratic polynomials in $\mathbb{F}_5[x]$.

(8) Construct a monic polynomial $f \in \mathbb{Z}[x]$ satisfying:

$$f \equiv \begin{cases} \text{(irreducible cubic)} & \text{mod } 2 \\ \text{(irreducible quadratic)(linear)} & \text{mod } 3 \end{cases}$$

in the rings $\mathbb{F}_2[x]$ and $\mathbb{F}_3[x]$ respectively. Determine the Galois group of f.

Do not forget to justify your answers!