THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

June 15, 2021 *3 hours*

Instructions: Answer as many questions as you can. Give complete justifications. Aim for complete solutions (there will be little or no partial credit). You should attempt at least two complex analysis questions and at least two real analysis questions.

---

1. Suppose f is a nonconstant function which is holomorphic on the closed unit disk

$$\bar{\mathbb{D}} = \{z \in \mathbb{C} | \ |z| \leq 1\}$$

and such that $|f(z)|$ is constant on $\partial\mathbb{D} = \{z \in \mathbb{C} | \ |z| = 1\}$.
Prove that f has a zero in $\mathbb{D} = \{z \in \mathbb{C} | \ |z| < 1\}$.

2. For

$$f(z) = \frac{z}{(z+1)(3-z)}$$

find the Laurent series (in powers of z)
(a) for $|z| < 1$; (b) for $1 < |z| < 3$; (c) for $|z| > 3$.

3. Suppose $D \subset \mathbb{C}$ is a domain, f is a function analytic in D, and at every $z \in D$ $f(z) = 0$ or $f'(z) = 0$. Show that f is constant.

4. Use residues to evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^8}.$$

5. A point e in a convex set $C \subseteq \mathbb{R}^n$ is called *extreme point* if it cannot be represented as $e = (1-\lambda)x + \lambda y$ for some distinct $x, y \in C$ and $\lambda \in (0,1)$.

Let A be an $m \times n$ matrix with rank m and columns denoted by $A_1, \ldots, A_n \in \mathbb{R}^m$ and let $b \in \mathbb{R}^m$. Define the set $F := \{x \in \mathbb{R}^n \mid Ax = b, \;\; x_i \geq 0 \text{ for all } i = 1, \ldots, n\}$ and let $x^* \in F$.

Show that if the set of columns $\{A_j | x_j^* \neq 0\}$ is linearly independent, then x^* is an extreme point of F.

6. Give an example of a function $f : [0,1] \to [0,1]$ that is non-decreasing, $f(0) = 0$, $f(1) = 1$, and has zero derivative almost everywhere.

7. Show the following identity

$$\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \, dy = \sqrt{2\pi}.$$

8. Let $(\mathbb{R}, d_1)$ and $(\mathbb{R}, d_2)$ be metric spaces and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous surjective map such that $d_1(p,q) \leq d_2(f(p), f(q))$ for every pair of points $p, q \in \mathbb{R}$. If $(\mathbb{R}, d_1)$ is complete, must $(\mathbb{R}, d_2)$ be complete? Give a proof or a counterexample.