PhD Comprehensive Exam (Algebra)
Department of Mathematics
October 2024

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. Justify all your answers.
4. Answer questions in the spaces provided.
5. Skim the questions at the start so that you can focus on the ones you feel most confident about.
6. The exam is divided into four parts
   (a) Linear Algebra.
   (b) Group theory.
   (c) Rings and Modules.
   (d) Field theory.

   Solve at **least one** problem from each part. With that said, you should try to solve as many problems as possible.
7. Each problem is worth 10 points.
8. There are eight problems.

Linear Algebra

1. (10 points) Let A and B be complex 6×6 matrices. Suppose that B is nilpotent, A has minimal polynomial $(x-1)^2$ and the 1-eigenspace of A is the same as the 0-eigenspace of B. Show that $B^4 = 0$.

2. (10 points) Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear operator and let $f \in \mathbb{C}[x]$ a polynomial. Suppose that λ is an eigenvalue of $f(T)$. Prove or disprove: there is an eigenvalue μ of T so that $\lambda = f(\mu)$.

Groups

3. (10 points) Let $G = GL_2(\mathbb{F}_p)$ where p is a prime. Note that $|G| = (p-1)^2p(p+1)$. Show that all Sylow p-subgroups of G are conjugate to the subgroup generated by

$$C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Determine the number of Sylow p-subgroups of G. [Hint: Consider the subgroup of upper triangular matrices.]

4. (10 points) Let G be a finite simple group. Let H be a subgroup of G of index $[G : H] = n$ with $n > 2$. Show that G is isomorphic to a subgroup of A_n where A_n is the alternating subgroup of S_n.

Rings and Modules

5. (10 points) Recall that an Artinian ring satisfies the descending chain condition on ideals. Show that if R is a commutative Artinian ring with identity, then any prime ideal of R is maximal.

6. (10 points) Let R be an integral domain and let M be a finitely generated module over R. Prove or disprove: if N is a submodule of M then N is finitely generated.

Fields

7. (10 points) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 4. Assume that it has 2 real roots and 2 complex roots. Show that this implies that the Galois group of $f(x)$ is either the dihedral group of order 8 or the symmetric group on 4 letters.

8. (10 points) Show that there is no $\alpha \in \mathbb{Q}(\sqrt[5]{3})$ with

$$\alpha^3 = 1 + \sqrt[5]{3}.$$