THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

October 8, 2024 *3 hours*

Instructions: There will be little or no partial credit, so you should aim to solve at least 5 problems completely and correctly rather than attempting every problem. Skim the questions at the start, so you can focus on the ones you feel most confident about. You should attempt at least two complex analysis questions and at least two real analysis questions.

Complex Analysis: Questions 1–4.

1. Let $\{f_k\}$, $k \in \mathbb{N}$, be a sequence of functions analytic in a region $R \subset \mathbb{C}$. Suppose that the series

$$F(z) = \sum_{k \geq 1} f_k(z)$$

converges uniformly on compact subsets of R. Prove that F is analytic in R.

2. Expand $f(z) = e^{z/(z-2)}$ in a Laurent series about $z = 2$ and determine the region of convergence. Classify the singularities of $f(z)$ at $z = 2$ and $z = \infty$.

3. Evaluate $\displaystyle\int_0^{2\pi} \frac{dt}{(5 - 3\sin(t))^2}$.

4. Use Rouché's theorem to prove the fundamental theorem of algebra.

Real Analysis: Questions 5–8.

5. Let (X, d) and (Y, ρ) be metric spaces and suppose $f : X \to Y$ is uniformly continuous. Prove or provide a counterexample: If $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in Y.

6. Find the volume of $\{(x, y, z) \in \mathbb{R}^3 : 0 \leq 2z \leq 1 \text{ and } (x^2 + y^2)(1 + x^2 + y^2)z \leq 1\}$.

7. Find a function $G(x, t)$ so that the solution to the boundary value problem $y''(x) = f(x)$ with $y(0) = y(1) = 0$ is $y(x) = \displaystyle\int_0^1 G(x, t) f(t)\, dt$ for all continuous functions f.

8. Let f and g be continuous, real-valued functions on $[0, 1]$ with $f(t) > 0$ for all t. Suppose that $\displaystyle\int_0^1 f(t)^n g(t)\, dt = 0$ for $n = 0, 1, 2, \ldots$. Prove that if $h : \mathbb{R} \to \mathbb{R}$ is also continuous, then $\displaystyle\int_0^1 h(f(t)) g(t)\, dt = 0$.