PhD Comprehensive Exam (Algebra)
Department of Mathematics
May 2024

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. Justify all your answers.
4. Answer questions in the spaces provided.
5. Skim the questions at the start so that you can focus on the ones you feel most confident about.
6. The exam is divided into four parts
   (a) Linear Algebra.
   (b) Group theory.
   (c) Rings and Modules.
   (d) Field theory.

   Solve at **least one** problem from each part. With that said, you should try to solve as many problems as possible.
7. Each problem is worth 10 points.
8. There are eight problems.

Linear Algebra

1. (10 points) Let F be a field. Prove that if $M \in M_{n\times n}(F)$ can be written in the form

$$\begin{bmatrix} A & B \\ O & C \end{bmatrix}$$

where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

2. (10 points) Let A be a 2×3 matrix with complex entries and let B be a 3×2 matrix with complex entries. Suppose that BA is similar to

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Prove that AB is similar to

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Groups

3. (10 points) Show that if a group G has a conjugacy class with two elements, then G is not simple.

4. (10 points) Let G be a group of order 66. Show that there is an $x \in G$ with $|x| = 33$.

Rings and Modules

5. (10 points) Suppose that A is an integral domain containing a subring K such that K is a field, and A is a finite dimensional vector space over K. Prove that A is a field.

6. (10 points) Let R be a principal ideal domain and let M be a non-zero R-module. Show that there is a submodule $N \subseteq M$ that is isomorphic to R/p where $p \subseteq R$ is a prime ideal. Note that M need not be finitely generated so you may not apply the classification theorem for finitely generated modules over a principal ideal domain.

   Fields

7. (10 points) Show that any finite extension of fields K/F is algebraic. Is the converse statement true? Make sure to carefully justify any claims made.

8. (10 points) Let $p \in \mathbb{Z}$ be a postive prime number. Consider the Galois extension of finite fields $\mathbb{F}_{p^2}/\mathbb{F}_p$. Let $\theta \in \mathbb{F}_{p^2}$ be a generator for the cyclic group $\mathbb{F}_{p^2}^{\times} := \mathbb{F}_{p^2} \setminus \{0\}$. Prove or disprove: the element $N_{\mathbb{F}_{p^2}/\mathbb{F}_p}(\theta)$ is a generator for the cyclic group $\mathbb{F}_p^{\times}$.

   Recall that $N_{\mathbb{F}_{p^2}/\mathbb{F}_p}$ is the field norm.