THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 9 2024, 1 - 4 PM

Instructions: Solve as many problems as you can. Carefully justify your answers. Aim for complete solutions (there will be little or no credit for partial answers).

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1. For $n = 1, 2, \ldots,$ let
$$\gamma_n = (1 + \frac{1}{2} + \cdots + \frac{1}{n}) - \ln n.$$
Show that $\lim_{n\to\infty} \gamma_n$ exists.

2. Show that the following limit exists:
$$\lim_{x\to\infty} \int_0^x \sin(t^2 - 1)dt.$$

3. Let $f : [0, 2\pi] \to \mathbb{C}$ be an *infinitely differentiable* function. For $n \in \mathbb{Z}$, let
$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx}dx$$
be the n-th *Fourier coefficient* of f. Show that $\hat{f}(n)$ is *rapid decay.* That is, for any positive integer k,
$$|n^k \hat{f}(n)| \to 0, \quad \text{as } n \to \infty.$$

4. Let $f : [0, 1] \to \mathbb{R}$ be a Riemann integrable function. Show that
$$\lim_{n\to\infty} \int_0^1 f(x)x^n dx = 0.$$

5. A complex-valued differentiable function $f(z)$ is called $\mathbb{C}$-differentiable at a point $z_0 \in \mathbb{C}$, if
$$\lim_{\mathbb{C}\ni h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$
exists. Find all points where the function $f(z) = |z|^2$ is $\mathbb{C}$-differentiable.

6. Show that if $f(z)$ is an entire function (i.e., holomorphic on $\mathbb{C}$) and there exists a nonempty disk D such that $f(z)$ does not attain any values in D, then f is constant.

7. Evaluate

$$\int_{|z|=1} \frac{e^z}{z^m}\, dz,$$

where $m \in \mathbb{Z}$.

8. Suppose that $f(z)$ is a holomorphic function in the unit disc in $\mathbb{C}$. Suppose that there exists a sequence of points $\{\zeta_j\}$, $\zeta_j \neq 0$, such that $\lim_{j\to\infty} \zeta_j = 0$ and $f(\zeta_j) = 0$ for all j. Prove that $f \equiv 0$.