PhD Comprehensive Exam (Algebra)
Department of Mathematics
May 2025

Instructions:

1. You have 3 hours to complete the exam.
2. Little partial credit will be given. Aim for complete solutions.
3. Justify all your answers.
4. Answer questions in the spaces provided.
5. Skim the questions at the start so that you can focus on the ones you feel most confident about.
6. The exam is divided into four parts
   (a) Linear Algebra.
   (b) Group theory.
   (c) Rings and Modules.
   (d) Field theory.

   Solve at **least one** problem from each part. With that said, you should try to solve as many problems as possible.
7. Each problem is worth 10 points.
8. There are eight problems.

Linear Algebra

1. (10 points) Let A be a real symmetric $n \times n$ matrix. Show there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors for A. Deduce that A is orthogonally diagonalizable.

2. (10 points) Let V be a finite dimensional $\mathbb{Q}$ vector space and let $T : V \to V$ be a linear operator. We say that a subspace $W \subseteq V$ is *invariant* if $T(W) \subseteq W$. We say that T is *irreducible* if the only invariant subspaces are V and the zero subspace.

   Show that if T is irreducible then the minimal polynomial of T is an irreducible polynomial in $\mathbb{Q}[x]$.

Groups

3. (10 points) Show that the symmetric group S_5 has 6 Sylow 5-subgroups.

   S_5 acts on its set of Sylow 5-subgroups P_5 by conjugation. Determine the homomorphism $\varphi : S_5 \to S_6$ induced by this group action on a set of generators for S_5.

   Use this to show that S_6 has 2 non-conjugate subgroups isomorphic to S_5.

   Hint: S_5 can be generated by a 5-cycle and a transposition.

4. (10 points) Let G be the group of upper triangular matrices in $GL_2(\mathbb{Z}/11\mathbb{Z})$. Let n_5 be the number of 5-Sylow subgroups of G. What is the value of n_5?

Rings and Modules

5. (10 points) Let F be a field and $R = M_{nn}(F)$ be the F-algebra of $n \times n$ matrices.

Show that $R = \oplus_{i=1}^{n} L_i$ is a direct sum of n simple left ideals L_i. For each $i \geq 2$, show that $L_i \cong L_1$ as R-modules.

Moreover, show that any simple R-module M is isomorphic to L_1 as an R-module.

Hint: Consider the matrices $E_{ij} \in M_{nn}(F)$ with a 1 in the (i,j) position and 0 in all other entries.

6. (10 points) Let R be a commutative ring with $0 \neq 1$. Let M be an R-module. For $m \in M$ we write $\mathrm{Ann}_R(m) \subseteq R$ for the annihilator ideal of m.

   Suppose that for every $m', m \in M$ with $m', m \neq 0$ we have that $\mathrm{Ann}_R(m) = \mathrm{Ann}_R(m')$.

   Prove or disprove: For a non-zero $m \in M$ the annihilator ideal $\mathrm{Ann}_R(m)$ is prime.

   Fields

7. (10 points) Let $K = \mathbb{Q}(\sqrt[8]{2}, i)$ be the splitting field for $f(x) = x^8 - 2 \in \mathbb{Q}[x]$ inside $\mathbb{C}$. Let $\theta_8 = e^{\frac{2\pi i}{8}}$.

   The Galois group of $K/\mathbb{Q}$ is generated by $\sigma \in \mathrm{Gal}(K/\mathbb{Q}(i))$ with $\sigma(\sqrt[8]{2}) = \sqrt[8]{2}\theta_8$ and τ which is the restriction of complex conjugation to K. The Galois group has presentation

   $$\langle \sigma, \tau : \sigma^8 = 1, \tau^2 = 1, \tau\sigma\tau^{-1} = \sigma^3 \rangle$$

   Show that $\mathbb{Q}(\sqrt{2}i)/\mathbb{Q}$ is a Galois subextension of $K/\mathbb{Q}$ and $K/\mathbb{Q}(\sqrt{2}i)$ has Galois group isomorphic to Q_8.

8. (10 points) Prove or disprove: $\mathbb{Q}(\sqrt[3]{2})$ is a subfield of some cyclotomic field over $\mathbb{Q}$.