THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

Ph.D. Comprehensive Examination (Analysis)

May 12, 2025 *3 hours*

Instructions: There will be little or no partial credit, so you should aim to solve at least 5 problems completely and correctly rather than attempting every problem. Skim the questions at the start, so you can focus on the ones you feel most confident about. You should attempt at least two complex analysis questions and at least two real analysis questions.

Complex Analysis: Questions 1–4.

1. Let A be the annulus
$$A = \{z \in \mathbb{C} | \; 1 \leq |z| \leq \sqrt{2}\}.$$
Suppose f is analytic on A, $|f(z)| \leq 1$ for all z on the circle $|z| = 1$ and $|f(z)| \leq 2\sqrt{2}$ for all z on the circle $|z| = \sqrt{2}$. Show: $|f(z)| \leq |z|^3$ in A.

2. Let $f(z) = e^{\overline{z}}$. Prove: there is no sequence $\{p_n(z)\}$ of polynomials in z that converges to f uniformly on the set $\{z \in \mathbb{C} : |z| = 1\}$.

3. Let $f = u + iv$ be an entire function (here $u = Re(f)$ and $v = Im(f)$). Suppose $|u(x, y)| - |v(x, y)| < 0$ for all (x, y). Show that f is constant.

4. Let $B(z_0; r)$ be the closed disk of radius $r > 0$ centered at $z_0 \in \mathbb{C}$. Suppose f is analytic on $B(z_0; r)$, $f(z_0) = z_0$, $f'(z_0) = 0$, and $M > 0$ is such that $|f''(z)| \leq M$ for all $z \in B(z_0; r)$. Prove:
$$|f(z) - z_0| \leq \frac{1}{2} M |z - z_0|^2$$
for any $z \in B(z_0; r)$.

Real Analysis: Questions 5–8.

5. Let (X, d) be a metric space and $x_0 \in X$. Suppose that for all $x \in X$ and all $r > 0$ there exists a continuous $f : [0, 1] \to X$ such that $f(0) = x_0$ and $f(1)$ is in the ball of radius r centred at x. Prove that X is connected. (Hint: Do not attempt to prove that X is path connected, it may not be.)

6. Find, with proof, the range of the function $f(x, y) = (x + y + 1)/(x^2 + y^2 + 24)$ on $\mathbb{R}^2$.

7. Let $\mathbb{R}[x]$ be the set of all polynomials in x with real coefficients. Fix $y \in \mathbb{R}$. If $r \in \mathbb{R}[x]$ is not the zero polynomial, define $\nu(r)$ to be the unique integer $m \geq 0$ such that $r(x) = (x-y)^m g(x)$ for some $g \in \mathbb{R}[x]$ such that $g(y) \neq 0$. Prove that $d(p,q)$ defined by

$$d(p,q) = \begin{cases} 2^{-\nu(p-q)}, & p \neq q; \\ 0, & p = q, \end{cases}$$

is a metric on $\mathbb{R}[x]$.

8. Fix $x_0 \in \mathbb{R}$. We say $f : \mathbb{R} \to \mathbb{R}$ is *upper semi-continuous at* x_0 if for all $y > f(x_0)$ there exists a $\delta > 0$ such that if $|x - x_0| < \delta$ then $f(x) < y$.

Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is upper semi-continuous at x_0 for $n = 1, 2, \ldots$. Also suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$ to a function f. Prove that f is upper semi-continuous at x_0.